When is rounding allowed?
A new approach to integer nonlinear optimization

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Abstract—In this paper we present a new approach for solving unrestricted integer nonlinear programming problems. More precisely, we investigate in which cases one can solve the integer version of a nonlinear optimization problem by rounding (up or down) the components of a solution for its continuous relaxation. The idea is based on the level sets of the objective function. We are able to identify geometric properties of the level sets that ensure that such a rounding property holds. We illustrate that such properties of level sets can be found in typical problems of location theory.

Index Terms—Level set approach, nonlinear integer optimization, continuous relaxation, location problems.

I. INTEGER NONLINEAR PROGRAMMING

It is well known that adding integrality constraints to continuous optimization problems in the majority of cases increases their complexity. A prominent example is linear programming, which can be solved in polynomial time by interior point methods, but gets NP hard if integrality constraints are added. Integer linear programming has been widely studied and been developed into a mature discipline. However, integer nonlinear programming is even harder than integer linear programming and the theory developed in this area is much less mature. According to [7] who provide a recent overview, “integer nonlinear programming is still a very young field”.

Integer nonlinear programming has been tackled by different communities. In the context of global optimization the main focus is to develop numerical procedures for solving nonlinear integer problems. Nowadays, also techniques from integer linear programming are transferred to nonlinear integer programs. Results exist for integer concave minimization (equivalently integer convex maximization) which are based on the observation that a (quasi-)concave function attains its minimum (if it exists) at an extreme point of the feasible set. Hence, an enumeration of all vertices of the convex hull of the integer points would solve the problem. More efficient structures based on total unimodularity for linear constraints allow polynomial procedures. Results for integer concave minimization can be found e.g. in [11]. The methods used for convex integer minimization are different; early approaches include the extension of branch & bound methods for linear integer programming [5] or an extension of the cutting plane technique of Kelley [9]. More recently, outer approximation procedures have been suggested [2]. Research has also been done for integer minimization of polynomial functions over polyhedral sets where an FPTAS is possible [10]. Boolean integer nonlinear optimization has
been considered for special types of problems, many of them motivated by discrete or network optimization as e.g. the quadratic assignment problem. An example for a recent approach can be found in [3].

In this paper we suggest a new approach for integer nonlinear optimization, namely to use the level sets of the objective function of the optimization problem. To this end we need some definitions.

Let a function \( f : \mathbb{R}^n \to \mathbb{R} \) be given. The level set of \( f \) with respect to some level \( z \in \mathbb{R} \) is defined as \( L_z(f) = \{ x \in \mathbb{R}^n : f(x) \leq z \} \). Using level sets, the optimization problem \( \min \{ f(x) : x \in F \} \) for some function \( f : \mathbb{R}^n \to \mathbb{R} \) and some set \( F \subseteq \mathbb{R}^n \) can be reformulated as \( \min \{ z : L_z(f) \cap F \neq \emptyset, z \in \mathbb{R} \} \), i.e. the goal is to identify the smallest level for which a feasible point exists. This approach is known as graphical approach and e.g. used to illustrate the solution of linear programs in \( \mathbb{R}^2 \).

Our idea now is to treat the integrality constraint for an optimization problem \( \min \{ f(x) : x \in \mathbb{Z}^n \} \) in the same way, i.e. to identify the smallest value \( z \in \mathbb{R} \) for which \( L_z(f) \cap \mathbb{Z}^n \neq \emptyset \). This approach provides structural insight into the properties of an optimal solution. In particular we will investigate the rounding property, i.e. in which cases one can solve the integer version of a nonlinear optimization problem by rounding (up or down) the components of a solution for its continuous relaxation. Depending on the geometric structure of the level set we will be able to better understand the integer version of the optimization problem and to derive new algorithmic approaches.

The remainder of the paper is structured as follows. In the next section we will introduce the rounding property before we present some examples from location theory in which such a property holds in Section III. In Section IV and Section V we present two different geometric criteria which ensure the rounding property. Section VI illustrates our concept and its application on some optimization problems also coming back to the location examples from Section III. Extensions of the concept are sketched in Section VII. The paper is ended by some conclusions and further research questions.

II. THE ROUNдинG PROPERTY

We consider integer nonlinear unrestricted optimization problems of the following type

\[
(IP) \quad \min \{ f(x) : x \in \mathbb{Z}^n \}.
\]

The continuous relaxation of \( (IP) \) is given as

\[
(CP) \quad \min \{ f(x) : x \in \mathbb{R}^n \}.
\]

Let us assume that an optimal solution \( x^* \) to the continuous relaxation \( (CP) \) of an integer program \( (IP) \) is known. The question we are dealing with is the following: In which cases does rounding this optimal solution \( x^* \) yield an optimal solution for \( (IP) \)? By rounding we mean to round each coordinate of \( x^* \) up or down to the respective next integer, i.e. for \( x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n \) we define

\[
\text{Round}(x) = \{ y \in \mathbb{Z}^n : y_i \in \{ \lfloor x_i \rfloor, \lceil x_i \rceil \} \forall i \}
\]

as the set of integer points with rounded coordinates. For \( x \in \mathbb{Z}^n \) we have \( \text{Round}(x) = \{ x \} \) and for \( x \in \mathbb{R}^n \), \( \text{Round}(x) \) contains at most \( 2^n \) points. We will also investigate when rounding \( x^* \) to its closest integer point yields an optimal solution for \( (IP) \). We therefore define \( \text{round}(x^*) \) to be the closest integer point to \( x^* \), i.e. the point whose coordinates \( x^*_i \) are rounded to the closest integer \( x_i \) for all \( i = 1, \ldots, n \) using the round half up rule in order to break ties.

We can now introduce the following two rounding properties. As we do not consider any constraints these properties will only depend on the objective function \( f \).

\begin{definition}
We say that a function \( f : \mathbb{R}^n \to \mathbb{R} \) has the rounding property if the following holds: For any optimal solution \( x^* \) to \( (CP) \) \( \min \{ f(x) : x \in \mathbb{R}^n \} \) there exists an optimal solution \( x' \) of \( (IP) \) \( \min \{ f(x) : x \in \mathbb{Z}^n \} \) such that

\[ x' \in \text{Round}(x^*). \]
\end{definition}
If the rounding property (1) holds it guarantees the following: If \((CP)\) is polynomially solvable then \((IP)\) is also solvable in polynomial time for any fixed dimension, namely by first solving \((CP)\) and then testing the at most \(2^n\) solutions in \(\text{Round}(x^*)\). This approach yields an efficient algorithm if
- problem \((CP)\) can be solved efficiently, and
- the dimension \(n\) is rather small.

Both assumptions are satisfied for many location problems in the plane. In the next section we will show that for example the Weber problem with rectangular distance as well as the problem of determining the centroid satisfy the rounding property such that their integer versions can be solved efficiently.

Note that the rounding property (1) is trivially satisfied, but not at all helpful for the special case of Boolean optimization problems

\[(BP) \quad \min \{ f(x) : x \in \{0,1\}^n \} \]

for \(f : [0,1]^n \to \mathbb{R}\), since enumerating all \(2^n\) possible 0/1 vectors is always an (inefficient) option of solving the problem.

In order to tackle also problems of type \((BP)\), and to be more efficient when solving problems of type \((IP)\) we introduce the strong rounding property.

**Definition 2.2:** We say that \(f : \mathbb{R}^n \to \mathbb{R}\) has the strong rounding property if the following holds: For any optimal solution \(x^*\) to \((CP)\) \(\min \{ f(x) : x \in \mathbb{R}^n \}\) there exists an optimal solution \(x'\) of \((IP)\) \(\min \{ f(x) : x \in \mathbb{Z}^n \}\) such that

\[x' = \text{round}(x^*).\] (2)

This guarantees the following: If \((CP)\) is polynomially solvable then \((IP)\) is also solvable in polynomial time.

Note that a function can have the rounding property (1) without being continuous as continuity is not required in our definition of the rounding property. However, continuity is usually helpful in order to solve \((CP)\).

### III. The Rounding Property for Integer Location Problems

As a first example let us consider the classical Weber problem with Euclidean distance \(\|\cdot\|_2\) in the plane [14], [6], [13], [4].

The problem can be described as follows: Given a set of existing facilities

\[A = \{A_1, \ldots, A_M\} \subseteq \mathbb{R}^2\]

with nonnegative weights \(w_m \geq 0\), find a new location \(X \in \mathbb{R}^2\) such that

\[f(X) = \sum_{m=1}^{M} w_m \|A_m - X\|_2\]

is minimized. The resulting optimal point hence minimizes the weighted sum of Euclidean distances to the existing facilities \(A_1, \ldots, A_M\). The problem has applications for example when a new facility (e.g. a warehouse) is to be built that communicates with all the existing facilities. Its integer version is \(\min \{ f(X) : X \in \mathbb{Z}^2 \}\).

Does the objective function of the integer Weber problem have the rounding property (1)? The answer is no.

To see this we consider the example in Figure 1 with two existing facilities \(A_1, A_2\) and weights \(w_1 = w_2 = 1\). Obviously the line \(\lambda A_1 + (1 - \lambda) A_2\) with \(\lambda \in [0,1]\) is optimal for the continuous problem. But for the point \(\hat{x}\) we see that none of the four points in \(\text{Round}(\hat{x})\) is optimal for the integer problem since \(f(x) < f(y) \forall y \in \text{Round}(\hat{x})\).

![Fig. 1. Counterexample that shows that in general the classical Weber problem with Euclidean distance does not have the rounding property.](image-url)
So we see that not even for such an easy location problem the rounding property is trivially satisfied. Nevertheless we are going to show that there are location problems that have the rounding property.

We remark that in this example the rounding property does not hold although the objective function is convex. While convexity is not sufficient for the rounding property in $\mathbb{R}^n$ if $n \geq 2$ it is sufficient for $n = 1$:

**Lemma 3.1:** Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Then $f$ has the rounding property (1).

**Proof:** Let $x^\ast$ be a minimum of $(CP) \min \{ f(x) : x \in \mathbb{R} \}$ and let $x'$ be a minimum of its integer version $(IP) \min \{ f(x) : x \in \mathbb{Z} \}$. This yields that $f(x^\ast) \leq f(x')$. We want to show that there exists $\tilde{x} \in \{ \lfloor x^\ast \rfloor, \lceil x^\ast \rceil \}$ with $f(\tilde{x}) \leq f(x')$. Let without loss of generality $x' \leq x^\ast$ and define $\tilde{x} := \lfloor x^\ast \rfloor$. Then we have $x' \leq \tilde{x} \leq x^\ast$. Convexity yields that

$$f(\tilde{x}) = f(\lambda x' + (1-\lambda)x^\ast) \leq \lambda f(x') + (1-\lambda) f(x^\ast) \leq f(x')$$

hence the rounding property (1) holds. □

In our next example we consider again a location problem but we use the rectangular distance $\| \cdot \|_1$ instead of the Euclidean distance $\| \cdot \|_2$ and obtain the Weber problem with rectangular distance in which we look for some point $X \in \mathbb{R}^2$ minimizing $\sum_{m=1}^{M} w_m \| A_m - X \|_1$ (for given existing facilities $A_m \in \mathbb{R}^2$ and given positive weights $w_m$ as before). Its integer counterpart hence is

$$\min \{ \sum_{m=1}^{M} w_m \| A_m - X \|_1 : X \in \mathbb{Z}^2 \}.$$

Does the objective of the rectangular Weber problem have the rounding property (1)? In this case the answer is yes.

**Lemma 3.2:** The function $f : \mathbb{R}^2 \to \mathbb{R}$, $X \mapsto \sum_{m=1}^{M} w_m \| A_m - X \|_1$ has the rounding property (1).

**Proof:** Let $A_m = (a_m, b_m)$ for all $m = 1, \ldots, M$ and $X = (x, y)$. Then we can rewrite

$$\sum_{m=1}^{M} w_m \| A_m - X \|_1 = \sum_{m=1}^{M} w_m |a_m - x| + \sum_{m=1}^{M} w_m |b_m - y|$$

and solve two convex one-dimensional integer problems of type

$$\min \{ \sum_{m=1}^{M} w_m |a_m - x| : x \in \mathbb{Z} \}$$

as $f_1$ and $f_2$ are convex [12]. Due to Lemma 3.1 both one-dimensional problems have the rounding property (1); consequently also the Weber problem with rectangular distance has. □

We remark that this does not only hold in two dimensions but also for any dimension $n$.

Our third example considers the centroid of a set of existing facilities. To this end we minimize $\sum_{m=1}^{M} w_m \| A_m - X \|_2^2$. In its integer version we ask for the solution for

$$\min \{ \sum_{m=1}^{M} w_m \| A_m - X \|_2^2 : X \in \mathbb{Z}^2 \}.$$

Again, this problem’s objective has the rounding property (1) as the following lemma shows.

**Lemma 3.3:** The function $f : \mathbb{R}^2 \to \mathbb{R}$, $X \mapsto \sum_{m=1}^{M} w_m \| A_m - X \|_2^2$ has the rounding property (1).

**Proof:** Let $A_m = (a_m, b_m)$ for all $m = 1, \ldots, M$ and $X = (x, y)$. Then we can again separate the objective by rewriting

$$\sum_{m=1}^{M} w_m \| A_m - X \|_2^2 = \sum_{m=1}^{M} w_m (a_m - x)^2 + \sum_{m=1}^{M} w_m (b_m - y)^2$$

and are once more left with two convex (as $w_m \geq 0$) one-dimensional integer problems for which the result follows from Lemma 3.1. □
From the examples of this section one might suspect that the rounding property only holds in the case of separable convex functions. In the following sections we show that this is not the case. For example we are going to show that the function $f(x) = \|x\|_2$ has the rounding property and this function is not separable convex. We will also show that non-continuous functions may have the rounding property.

Although using examples from location theory for demonstrating the applicability of our concept the rounding property is certainly also interesting for any other type of nonlinear objective function with integrality constraint.

IV. CROSS-SHAPED LEVEL SETS

In geometry, one knows star-shaped sets: A set $M \subseteq \mathbb{R}^n$ is called star-shaped if a point $x_0 \in M$ exists such that for any $y \in M$ the line segment $\lambda x_0 + (1 - \lambda)y$ ($\lambda \in [0, 1]$) is contained in $M$ [1].

Boltyanski, Martini and Soltan [1] generalize this definition by introducing $d$-star-shaped sets for any norm $d : \mathbb{R}^n \to \mathbb{R}$ as follows:

For $a, b \in \mathbb{R}^n$ denote by $[a, b]_d = \{ x \in \mathbb{R}^n : d(a, x) + d(x, b) = d(a, b) \}$ the $d$-segment of $a$ and $b$ with respect to the norm $d$. Then a set $M \subseteq \mathbb{R}^n$ is called d-star-shaped if a point $x_0 \in M$ exists such that for any $y \in M$ the $d$-segment $[x_0, y]_d$ is contained in $M$.

One can easily see that $l_2$-star-shapedness is equivalent to star-shapedness as defined above. In the following we will denote $l_1$-star-shaped sets as cross-shaped. Furthermore we will specify the required point $x_0$ and talk of cross-shaped with respect to some point, see the example in Figure 2 - left.

Definition 4.1: A set $M \subseteq \mathbb{R}^n$ is called cross-shaped w.r.t. $x_0$ if for any $y \in M$ the $l_1$-segment $[x_0, y]_{l_1}$ is contained in $M$.

As the definition of $[x_0, y]_{l_1}$ is not very illustrative we indicate that $[a, b]_{l_1} := \{ p \in \mathbb{R}^n : p_i = \lambda_i a_i + (1 - \lambda_i)b_i, \lambda_i \in [0, 1] \forall i = 1, \ldots, n \}$

This fact can easily be checked [8]. Thus we can roughly say $[a, b]_{l_1} = [a_1, b_1] \times \ldots \times [a_n, b_n]$ where we mean $[a_i, b_i] = \begin{cases} [a_i, b_i] & \text{if } a_i \leq b_i \\ [b_i, a_i] & \text{else} \end{cases}$.

This means that you can imagine $[a, b]_{l_1}$ as a paraxial box given by $a$ and $b$.

Note that cross-shaped sets need not be convex (see Figure 2 - left) and convex sets need not be cross-shaped (see Figure 2 - right). On the other hand cross-shaped sets are always star-shaped but star-shaped sets need not be cross-shaped (see Figure 2-middle and right).

We are interested in cross-shaped sets because the fact that the level sets of a function are cross-shaped guarantees that the function has the rounding property.

Theorem 4.1: Assume that any optimal solution $x^*$ of $(CP)$ $\min \{ f(x) : x \in \mathbb{R}^n \}$, satisfies that the level sets $L_{\leq}(z)$ of the objective function $f : \mathbb{R}^n \to \mathbb{R}$ are cross-shaped w.r.t. $x^*$ for all $z \in \mathbb{R}$ with $f(x^*) \leq z \leq \bar{z} = \min \{ f(x) : x \in \text{Round}(x^*) \}$. Then $f$ has the rounding property (1).

Proof: Choose any optimal solution $x^*$ of $(CP)$. Let $x \in \mathbb{Z}^n$ be an arbitrary point with objective value $z := f(x)$ and $\bar{x} := \arg\min \{ f(x) : x \in \text{Round}(x^*) \}$. We aim at showing that $f(\bar{x}) \leq f(x)$.

Case 1: $\bar{z} \leq z \Rightarrow f(\bar{x}) = \bar{z} \leq f(x)$ and we are done.

Case 2: $\bar{z} > z$: In this case we know that $L_{\leq}(z)$ is cross-shaped w.r.t. $x^*$ and (of course) $x \in L_{\leq}(z)$. But this means $[x^*, x]_{l_1} \subseteq L_{\leq}(z)$. Define $y_i := \begin{cases} [x^*_i] & \text{if } x_i > x^*_i \\ [x^*_i] & \text{if } x_i < x^*_i \\ x^*_i & \text{if } x_i = x^*_i \end{cases}$

which is obviously in $\text{Round}(x^*)$ and satisfies

$$\sum_{i=1}^{n} |x^*_i - y_i| + \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x^*_i - y_i| + |x_i - y_i|.$$
In contrast the right figure shows an example for a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) whose level set \( L_{\leq}(\bar{z}) \) (for example) is not cross-shaped (as not connected) and that does not have the rounding property (1) as \( \text{Round}(x^*) = \{1, 2\} \cap \text{Opt}(IP) = \{x \in \mathbb{Z} : x \leq 0\} = \emptyset \).

As we already mentioned the rounding property (1) is not at all helpful for Boolean problems (BP) and is also not satisfying if \( n \) is too large. So we would like to know whether cross-shapedness also provides us with the strong rounding property (2). The answer is no - see the example in Figure 4. The problem is that the level sets do not have to grow in the direction towards \( \text{round}(x^*) \) to be cross-shaped. Hence we are interested in identifying cross-shaped sets that also guarantee the strong rounding property (2).

**Definition 4.2:** A set \( M \subseteq \mathbb{R}^n \) is called coordinate axially symmetric w.r.t. \( x \) if for any \( y \in M \) and for all \( i \in \{1, \ldots, n\} \) it holds that \( z^{(i)} \in \mathbb{R}^n \) with
\[
z^{(i)}_j := \begin{cases} y_j & \text{if } j \neq i \\ 2x_j - y_j & \text{if } j = i \end{cases}
\]
is in \( M \). Thus \( M \) is axially symmetric with respect to all axes through \( x \) and parallel to the coordinate axes.

As an example, consider Figure 2: The set on the left and the one in the middle are coordinate axially symmetric w.r.t \( x \) whereas the set on the right is not.

In combination with cross-shapedness this property guarantees the strong rounding property.

**Lemma 4.1:** Assume that for any optimal solution \( x^* \) of (CP) \( \min \{ f(x) : x \in \mathbb{R}^n \} \), it holds that the level sets \( L_{\leq}(z) \) of the objective function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) are cross-shaped and coordinate axially symmetric w.r.t. \( x^* \) for all \( z \in \mathbb{R} \) with \( f(x^*) \leq z \leq \bar{z} = \min \{ f(x) : x \in \text{Round}(x^*) \} \). Then \( f \) has the strong rounding property (2).

**Proof:** From Theorem 4.1 we already know that \( f \) has the rounding property. It remains to show that \( f(\text{round}(x^*)) = \bar{z} \).

Since \( \text{round}(x^*) \in \text{Round}(x^*) \) we have that

\[
\text{round}(x^*) \geq \bar{z}.
\]

(3)

\( L_{\leq}(\bar{z}) \) is cross-shaped and coordinate axially symmetric w.r.t. \( x^* \). Let \( \bar{x} = \operatorname{argmin}\{ f(x) : x \in \text{Round}(x^*) \} \), i.e. \( f(\bar{x}) = \bar{z} \). From the definition of \( \text{round}(x^*) \) we know that

\[
|\bar{x}_i - x^*_i| \geq |\text{round}(x^*)_i - x^*_i| \quad \forall i \leq n.
\]

(4)

Define \( y \in \mathbb{R}^n \) as

\[
y_i = \begin{cases} 
\bar{x}_i & \text{if } \text{round}(x^*)_i = \bar{x}_i \\
2x^*_i - \bar{x}_i & \text{else}
\end{cases}
\]

This means we gain \( y \) by repeated reflection of the coordinates of \( \bar{x} \) along the axes through \( x^* \) which are parallel to the coordinate axes. As \( L_{\leq}(\bar{z}) \) is coordinate axially symmetric w.r.t. \( x^* \) and \( \bar{x} \in L_{\leq}(\bar{z}) \) each of these reflections delivers a point that lies within \( L_{\leq}(\bar{z}) \). Thus \( y \in L_{\leq}(\bar{z}) \).

As \( L_{\leq}(\bar{z}) \) is cross-shaped w.r.t. \( x^* \) we know that \([x^*,y]_i \subseteq L_{\leq}(\bar{z})\).

Now we aim to show that \( \text{round}(x^*) \in [x^*,y]_i \), i.e. we want to show that

\[
\text{round}(x^*)_i \in \begin{cases} [x^*_i,y_i] & \text{if } x^*_i \leq y_i \\
[y_i,x^*_i] & \text{else}
\end{cases}
\]

for all \( i \in \{1, \ldots, n\} \).

If \( \text{round}(x^*)_i = \bar{x}_i \) we have \( \text{round}(x^*)_i = y_i \) and we are done.

If \( \text{round}(x^*)_i \neq \bar{x}_i \) then \( y_i = 2x^*_i - \bar{x}_i \).

In the case that \( x^*_i < \text{round}(x^*)_i \) we know that \( \bar{x}_i < x^*_i \) as both \( \text{round}(x^*) \) and \( \bar{x} \) are in \( \text{Round}(x^*) \). Together with (4) this gives us \( \text{round}(x^*)_i - x^*_i \leq x^*_i - \bar{x}_i \). Hence we have

\[
y_i = 2x^*_i - \bar{x}_i \geq \text{round}(x^*)_i - x^*_i + x^*_i,
\]

i.e. \( \text{round}(x^*)_i \in [x^*_i,y_i] \). Analogously we can show that for \( \text{round}(x^*)_i < x^*_i \) it holds that \( \text{round}(x^*)_i \in [y_i,x^*_i] \).

As \([x^*,y]_i \) is a subset of \( L_{\leq}(\bar{z}) \) this gives us \( f(\text{round}(x^*)) \leq \bar{z} \). Together with (3) we conclude \( f(\text{round}(x^*)) = \bar{z} \).

Now we are of course interested in identifying functions that satisfy the assumptions of Theorem 4.1. This means that their level
sets are cross-shaped w.r.t. $x^*$ for all $z \in \mathbb{R}$ that satisfy $f(x^*) \leq z \leq \bar{z} = \min \{ f(x) : x \in \text{Round}(x^*) \}$ and for any optimal solution $x^*$ of $(CP)$ $\min \{ f(x) : x \in \mathbb{R}^n \}$. One class of functions that fulfill these assumptions are the separable convex functions: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called separable convex if $f$ can be written as

$$f(x) = \sum_{i=1}^{n} f_i(x_i)$$

with convex functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$ for all $i = 1, \ldots, n$ [7].

Lemma 4.2: The level sets of a separable convex function $f$ are cross-shaped with respect to any optimal solution $x^*$ of (CP).

Proof: Let $x^*$ be an optimal solution for (CP). As $f$ is separable convex this means that $f_i(x_i^*) \leq f_i(x) \forall x \in \mathbb{R}$ and for all $i = 1, \ldots, n$.

Let $z \geq f(x^*)$. We want to show that this level set is cross-shaped w.r.t. $x^*$. If $L_{\leq}(z) = \{ x^* \}$ we are done. Therefore assume that there is another point $y \in L_{\leq}(z)$. Then every point $p \in [x^*, y]_{I_i}$ can be described as $p = (\lambda_1 x_i^* + (1 - \lambda_1) y_i, \ldots, \lambda_n x_n^* + (1 - \lambda_n) y_n)$ with $\lambda_i \in [0,1]$ for all $i = 1, \ldots, n$ (compare the remark after Definition 4.1). This yields

$$f(p) = \sum_{i=1}^{n} f_i(\lambda_i x_i^* + (1 - \lambda_i) y_i) \leq \sum_{i=1}^{n} \lambda_i f_i(x_i^*) + (1 - \lambda_i) f_i(y_i) \leq f(y) \leq z \quad \text{because } y \in L_{\leq}(z).$$

So we have $p \in L_{\leq}(z)$ and thus $[x^*, y]_{I_i} \subseteq L_{\leq}(z)$ for every $y \in L_{\leq}(z)$.

Summarizing, $L_{\leq}(z)$ is cross-shaped w.r.t. $x^*$ for all $z \geq f(x^*)$.

We remark that this is an alternative proof for Lemma 3.1.

It is important to remark that cross-shapedness is sufficient for the rounding property but surely not necessary. Therefore we will analyze a completely different geometric form in the following section which also guarantees the rounding property.

V. QUASI-ROUND LEVEL SETS

Let us first consider functions whose level sets are balls about an optimal solution $x^*$. I.e. for any level $z$ there exists $r_z \in \mathbb{R}^+$ such that

$$L_{\leq}(z) = \{ y : \| y - x^* \|_p \leq r_z \}.$$

It is rather obvious that such functions have even the strong rounding property (2). Indeed, this is exactly where our basic idea comes from: if you know that the level sets are (Euclidean) balls, it is intuitively clear that the first integer point you reach by blowing up the level sets is the one with the shortest Euclidean distance.

One can also show that this stays true if one uses a general $p$-norm instead of the Euclidean norm.

Lemma 5.1: A function $f$ whose level sets satisfy $L_{\leq}(z) = \{ y : \| y - x^* \|_p \leq r_z \}$ with some $r_z \in \mathbb{R}^+$ for all $z$ with $f(x^*) \leq z \leq \bar{z} = \min \{ f(x) : x \in \text{Round}(x^*) \}$ has the strong rounding property (2).

Proof: Let $x^*$ be a minimizer of (CP) and $x'$ be a minimizer of (IP), i.e. $x^* = \min \{ f(x) : x \in \mathbb{R}^n \}$ and $x' = \min \{ f(x) : x \in \mathbb{Z}^n \}$. We want to show that $\text{Round}(x^*)$ is as good as $x'$, i.e. $f(\text{Round}(x^*)) \leq f(x')$.

Obviously $f(x') \leq \bar{z}$ as $\text{Round}(x^*) \subseteq \mathbb{Z}^n$. But then $L_{\leq}(f(x')) = \{ y : \| y - x^* \|_p \leq r \}$ for some $r \in \mathbb{R}^+$ (see the assumption). As $x'$ is in $L_{\leq}(f(x'))$ it follows that $r \geq \| x' - x^* \|_p$.

Moreover, for $\text{Round}(x^*)$ it holds for all $x \in \mathbb{Z}^n$ that $\| (\text{Round}(x^*))_i - x^*_i \| = \| x_i - x^*_i \| \forall i = 1, \ldots, n$. This yields

$$\| \text{Round}(x^*) - x^* \|_p \leq \| x - x^* \|_p \quad \forall x \in \mathbb{Z}^n,$$

in particular we have $\| \text{Round}(x^*) - x^* \|_p \leq \| x' - x^* \|_p$. Therefore $\text{Round}(x^*) \in L_{\leq}(f(x'))$ and consequently $f(\text{Round}(x^*)) \leq f(x')$. 

In the following we show that we do not have to be this restrictive, but can generalize this
property even further. To this end let us denote the unit ball about $x$ with respect to the norm \( \| \cdot \|_p \) and radius $r$ by

\[ B_{r,p}(x) = \{ y \in \mathbb{R}^n : \| x - y \|_p \leq r \}. \]

With $d_p(x, M)$ we denote the $\| \cdot \|_p$-norm distance between $x$ and a set $M \subseteq \mathbb{R}^n$, i.e.

\[ d_p(x, M) = \min_{y \in M} \| x - y \|_p. \]

**Definition 5.1:** Given $\alpha \geq 0$ and $p \geq 1$, we call a set $M \subseteq \mathbb{R}^n (\alpha, p)$-quasiround w.r.t. a center $x_0$ if there exist radii $r, R \in \mathbb{R}_0^+$ such that $B_{r,p}(x_0) \subseteq M \subseteq B_{R,p}(x_0)$ and $R - r \leq \alpha$.

Note that quasiround sets need not be connected and that it depends on the given $p$ whether a set is quasiround or not (for an illustration see Figure 5).

As mentioned before we want to show that quasiroundness (just as cross-shapedness in the previous section) provides us with the rounding property.

**Theorem 5.1:** Assume that any optimal solution $x^*$ of (CP) \( \min \{ f(x) : x \in \mathbb{R}^n \} \) satisfies that the level sets $L_\leq(z)$ of the objective function $f : \mathbb{R}^n \to \mathbb{R}$ are $(\alpha_p, p)$-quasiround with center $x^*$ and with respect to

\[ \alpha_p = \min_{y \in \mathbb{R}^n} \{ d_p(y, \mathbb{Z}^n \setminus \text{Round}(y)) - d_p(y, \mathbb{Z}^n) \} \]

for all $z \in \mathbb{R}$ with $f(x^*) \leq z \leq \tilde{z} = \min \{ f(x) : x \in \text{Round}(x^*) \}$. Then $f$ has the rounding property (1).

We remark that $\alpha_p \geq 0$.

**Proof:** Choose any optimal solution $x^*$ of (CP). Let $x \in \mathbb{Z}^n$ be an arbitrary point with objective value $\tilde{z} := f(x)$ and $\bar{x} := \arg\min \{ f(x) : x \in \text{Round}(x^*) \}$. Our goal is to show that $f(\bar{x}) \leq f(x)$.

Case 1: $\tilde{z} \leq z \Rightarrow f(\bar{x}) \leq f(x)$ and we are done, or:

Case 2: $\tilde{z} > z$: Then we have

1. $x \notin \text{Round}(x^*)$ and
2. there are $r \in \mathbb{R}_0^+$, $R \in \mathbb{R}^+$ with $B_p(x^*, r) \subseteq L_\leq(z) \subseteq B_p(x^*, R)$ and $R - r \leq \alpha_p$.

Let $x' \in \text{argmin} \{ \| x^* - y \|_p : y \in \mathbb{Z}^n \}$, i.e. $\| x^* - x' \|_p \leq d_p(x^*, \mathbb{Z}^n)$. From Lemma 5.1 we know that we can choose $x' = \text{round}(x^*)$. By definition of $\alpha_p$ we hence obtain

\[ \alpha_p \leq d_p(x^*, \mathbb{Z}^n \setminus \text{Round}(x^*)) - d_p(x^*, \mathbb{Z}^n) \]

\[ \leq d_p(x^*, \mathbb{Z}^n \setminus \text{Round}(x^*)) - \| x^* - x' \|_p. \]

Then

\[ \| x^* - x' \|_p \leq d_p(x^*, \mathbb{Z}^n \setminus \text{Round}(x^*)) - \alpha_p \]

\[ \leq \| x^* - x \|_p - \alpha_p \]

as $x \in \mathbb{Z}^n \setminus \text{Round}(x^*)$.

\[ \leq R - \alpha_p \]

as $x \in L_\leq(z) \subseteq B_p(x^*, R)$.

\[ \leq r. \]

Therefore $x' \in B_p(x^*, r) \subseteq L_\leq(z)$ and thus $f(x') \leq z < \tilde{z}$. But as $x' \in \text{Round}(x^*)$ we also have $\tilde{z} = f(\bar{x}) \leq f(x')$. Together we obtain $\tilde{z} \leq f(x') \leq z < \tilde{z}$, a contradiction.

Obviously this works mainly because we have chosen $\alpha_p$ the way it “fits”. To use this theorem it is therefore very important to know the size of $\alpha_p$.

First we remark that $\alpha_p > 0$ for all $p \geq 1$.

**Lemma 5.2:** For $p \geq 1$ it holds that $\alpha_p > 0$.

**Proof:** Obviously we have for all $x \in \mathbb{R}^n$

\[ d_p(x, \mathbb{Z}^n \setminus \text{Round}(x^*)) \geq d_p(x, \mathbb{Z}^n). \]

Thus $\alpha_p \geq 0$.

Assume that there is $x_0 \in \mathbb{R}^n$ with

\[ d_p(x_0, \mathbb{Z}^n \setminus \text{Round}(x_0)) = d_p(x_0, \mathbb{Z}^n). \]

Let

\[ y = \arg\min_{x \in \mathbb{Z}^n \setminus \text{Round}(x_0)} \| x_0 - x \|_p. \]

Define

\[ \bar{y} = \begin{cases} \lfloor x_0 \rfloor_i & \text{if } y_i > (x_0)_i; \\ \lfloor x_0 \rfloor_i & \text{if } y_i < (x_0)_i; \\ (x_0)_i & \text{if } y_i = (x_0)_i. \end{cases} \]

Then we have that $\bar{y} \in \text{Round}(x_0)$ and furthermore $\| \bar{y} - x_0 \|_p < \| y - x_0 \|_p$. As shown in
Lemma 5.1 there is an optimal solution \( x' \) to of (IP) in \( \text{Round}(x_0) \). Therefore

\[
\|x_0 - x'\|_p = d_p(x_0, \mathbb{Z}^n) \\
= d_p(x_0, \mathbb{Z}^n \setminus \text{Round}(x_0)) \\
= \|x_0 - y\|_p \\
> \|x_0 - \bar{y}\|_p \\
\geq \|x_0 - x'\|_p
\]

I.e. there can not be an \( x_0 \in \mathbb{R}^n \) that fulfills \( d_p(x_0, \mathbb{Z}_n \setminus \text{Round}(x_0)) = d_p(x_0, \mathbb{Z}^n) \) and thus \( \alpha_p > 0 \).

Now we really want to determine the size of \( \alpha_p \). This leads to the following optimization problem

\[
(P_p) \quad \min \left( \min_{y \in \mathbb{Z}^n \setminus \text{Round}(x)} \{\|x - y\|_p\} \right) \\
- \min_{y \in \mathbb{Z}^n} \{\|x - y\|_p\} \right) \\
\text{s.t.} \quad x \in \mathbb{R}^n
\]

for every \( 1 \leq p \leq \infty \). (The optimal objective value provides us with \( \alpha_p \).

**Theorem 5.2:** For \( p \in \mathbb{N} \) it holds that \( \alpha_p = \min\{\alpha'_p, 1\} \) with \( \alpha'_p \) the optimal objective value

Using Theorem 5.2 we can explicitly solve \((P_p)\) for \( p = 1 \) and \( p = 2 \).

**Lemma 5.3:** An optimal solution for \((P_2)\) is \( x = (x_1, \ldots, x_n) \) with \( x_i \in \mathbb{Z} \) for one \( i \in \{1, \ldots, n\} \) and \( x_j = \lfloor x_j \rfloor + 0.5 \) for all \( j \in \{1, \ldots, n\} \setminus \{i\} \). The optimal objective value is \( \alpha_2 = 0.5 \left( \sqrt{n+3} - \sqrt{n-1} \right) \).

**Proof:** For \( p = 2 \) the problem \((P_p')\) becomes

\[
(P_2') \quad \min \left( \sum_{i=1}^{n} a_i^2 + 1 + 2a_1 - \sum_{i=1}^{n} a_i^2 \\
\text{s.t.} \quad a_1 \leq a_j \quad \forall j \in \{2, \ldots, n\} \\
0 \leq a_i \leq 0.5 \quad \forall i \in \{1, \ldots, n\}.
\]

This problem can be solved as follows: The feasible set \( A := \{(a_1, \ldots, a_n) : 0 \leq a_i \leq 0.5 \ \forall i, \ a_1 \leq a_j \ \forall j \in \{2, \ldots, n\}\} \) is a closed and bounded subset of \( \mathbb{R}^n \) and therefore compact. As \( f : A \rightarrow \mathbb{R}, \ (a_1, \ldots, a_n) \mapsto \sqrt{\sum_{i=1}^{n} a_i^2 + 1 + 2a_1 - \sqrt{\sum_{i=1}^{n} a_i^2}} \) is continuous it must attain its maximum and minimum.
on the set $A$. Note that $f$ is differentiable for $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$, i.e., in the interior of $A$.

A necessary condition for a point $a^*$ in the interior of $A$ to be a minimum of $f$ is that the gradient of $f$ vanishes at $a^*$. It can be checked (by calculating the gradient, see [8]) that this is never the case. So we have to search the boundary - i.e. we fix one of the $a_i$'s and consider the lower-dimensional problem. Again there is no point where the gradient vanishes and again we have to search the boundary - i.e. we fix the next $a_i$. By induction we gain that a minimum lies in $\{0, 0.5\}^n$. By comparison we see that the minimum of $f$ is attained for $a_1 = 0$ and $a_j = 0.5$ for $j \in \{2, \ldots, n\}$.

This provides us with the objective value
\[
\alpha_2' = \sqrt{\sum_{i=1}^{n} a_i^2 + 1 + 2a_1} - \sqrt{\sum_{i=1}^{n} a_i^2} \\
= \sqrt{\sum_{i=2}^{n} 0.25 + 1} - \sqrt{\sum_{i=2}^{n} 0.25} \\
= \sqrt{0.25(n-1+4) - 0.25(n-1)} \\
= 0.5\sqrt{n+3 - \sqrt{n-1}}.
\]

Finally (see Theorem 5.2) we can find $\alpha_2$ as
\[
\alpha_2 = \min \{ \alpha_2', 1 \} \\
= \min \{ 0.5\sqrt{n+3 - \sqrt{n-1}}, 1 \} \\
= 0.5\sqrt{n+3 - \sqrt{n-1}}.
\]

Intuitively, this result could have also been guessed by the following consideration: to minimize $f$ we want to minimize the difference between $\sum_{i=1}^{n} a_i^2 + 1 + 2a_1$ and $\sum_{i=1}^{n} a_i^2$. Thus we want to minimize $1 + 2a_1$ and so we choose $a_1 = 0$. Then we want to make $\sum_{i=1}^{n} a_i^2$ as big as possible to “minimize the influence of the addend 1”: thus we choose $a_j = 0.5$ for all $j = 2, \ldots, n$.

Lemma 5.4: For $p = 1$ it holds that
\[
d_1(x, \mathbb{Z}^n \setminus \text{Round}(x)) - d_1(x, \mathbb{Z}^n) = 1
\]
for all $x \in \mathbb{R}^n$ and therefore $\alpha_1 = 1$.

Proof: For $p = 1$ the problem $(P_p')$ becomes
\[
(P_p') \min \sum_{i=1}^{n} a_i + 1 - \sum_{i=1}^{n} a_i \\
\text{s.t. } \alpha_j \forall j \in \{2, \ldots, n\} \\
0 \leq a_i \leq 0.5 \forall i \in \{1, \ldots, n\},
\]

hence $a_1 = 1$.

For $p = \infty$ the original problem can be solved directly.

Lemma 5.5: An optimal solution for $(P_{\infty})$ is $x = (x_1, \ldots, x_n)$ with $x_i = \lceil x_i \rceil + 0.5$ for one $i \in \{1, \ldots, n\}$ and $x_j \in \mathbb{Z}$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. The optimal objective value is $\alpha_{\infty} = 0.5$.

Proof: We will show that $\alpha_{\infty} \geq 0.5$ and that $\alpha_{\infty} \leq 0.5$.

1: $\alpha_{\infty} \geq 0.5$: We have that
\[
\min_{x \in \mathbb{R}^n} \{ d_\infty(x, \mathbb{Z}^n \setminus \text{Round}(x)) - d_\infty(x, \mathbb{Z}^n) \} \\
\geq \min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{Z}^n \setminus \text{Round}(x)} \| y - x \|_{\infty} \\
- \max_{x \in \mathbb{R}^n} \{ \min_{y \in \mathbb{Z}^n} \| y - x \|_{\infty} \}.
\]

Assume that $\| x - y \|_{\infty} < 1$ which is equivalent to $\max_{i \in \{1, \ldots, n\}} | x_i - y_i | < 1$ and this implies by definition that $y \in \text{Round}(x)$, hence
\[
\min_{x \in \mathbb{R}^n} \{ \min_{y \in \mathbb{Z}^n \setminus \text{Round}(x)} \| y - x \|_{\infty} \} \geq 1.
\]

On the other hand,
\[
\max_{x \in \mathbb{R}^n} \{ \min_{y \in \mathbb{Z}^n} \| y - x \|_{\infty} \} \leq 0.5
\]

since for every $x \in \mathbb{R}^n$ there is a $y^{(x)} \in \mathbb{Z}^n$ with
\[
y^{(x)}_i = \begin{cases} 
[x_i] & \text{if } x_i - [x_i] \leq [x_i] - x_i \\
[x_i] & \text{else},
\end{cases}
\]
hence $| x_i - y^{(x)}_i | = \min \{ x_i - [x_i], [x_i] - x_i \} \leq 0.5$ for all $i$ and therefore $\| x - y^{(x)} \|_{\infty} \leq 0.5$.

Together $\alpha \geq 1 - 0.5 = 0.5$.

2: $\alpha_{\infty} \leq 0.5$: Consider for example $x = (0.5, 1, \ldots, 1)$. Then we can compute
\[
d_\infty(x, \mathbb{Z}^n \setminus \text{Round}(x)) - d_\infty(x, \mathbb{Z}^n) = 0.5.
\]
Thus $\alpha_{\infty} \leq 0.5$.

Another interesting question is to ask in which cases the strong rounding property holds. Do functions with quasiround level sets have the strong rounding property (2)? The answer is no, as the following example (see Figure 6) shows: Let $x^* = (0.8, 0.8)$, i.e. $\text{round}(x^*) = (1, 1)$. The level set depicted (for one special $z$) is given by

$$L_\leq(z) = \{x \in \mathbb{R}^2 : ||x - x^*||_2 \leq 0.25 \ \lor \ 2/3 \leq ||x - x^*||_2 \leq 0.85\}.$$

Then $L_\leq(z)$ is quasiround as $0.85 - 0.25 = 0.6 < 0.5 \cdot (\sqrt{5} - 1) = \alpha_2$ (see Lemma 5.3 for $n = 2$) and one can easily think of a sequence of level sets for smaller $z$ that have the same form and are thus also quasiround.

Nevertheless $x_1$ and $x_3$ have an objective value smaller or equal to $z$ and $f(\text{round}(x^*)) > z$, as $\text{round}(x^*) \notin L_\leq(z)$. Thus $x^*_IP \in \{x_1, x_3\}$.

Fig. 6. Example of a function whose level sets are quasiround and that doesn’t have the strong rounding property.

Note that this level set is also axially symmetric w.r.t. $x^*$ (and the sequence of level sets for smaller $z$ can also be chosen this way) but nevertheless the function does not have the strong rounding property.

VI. ILLUSTRATIONS

After finding these two geometric forms (cross-shapedness and quasiroundness) that guarantee the rounding property we are interested in finding examples of functions whose level sets have these forms. Therefore we are going to apply them to the location problems of Section III.

As shown in Section III the objective of the Weber problem with Euclidean distance does not have the rounding property. But it is easy to see that the counterexample in Figure 1 does not contradict our theory: Consider the two given points $A_1$ and $A_2$ in Figure 1 with

- $(A_1)_1 < (A_2)_1$
- $(A_1)_2 = 3$ and $(A_2)_2 = 1$.

Then the level set to the optimal objective value $f(x^*) = ||A_1 - A_2||_2$ is given by

$$L_\leq(f(x^*)) = \{\lambda A_1 + (1 - \lambda) A_2; \lambda \in [0, 1]\}$$

and is neither cross-shaped nor quasiround:

- It cannot be cross-shaped w.r.t. any point $x_0 \in L_\leq(0)$ as for any point $y \in L_\leq(0)$ with $y \neq x_0$ we have that $((x_0)_1, (x_0)_2) \notin L_\leq(0)$.
- It cannot be quasiround w.r.t. any $x_0 \in L_\leq(0)$ as only for $r = 0$ the ball $B_{r,p}(x_0)$ is contained in $L_\leq(0)$ and

$$R \geq 0.5 \cdot ||A_1 - A_2||_2 = 0.5 \cdot \sqrt{(A_1)_1 - (A_2)_1^2 + 4} \geq 0.5 \cdot \sqrt{2} = \sqrt{2}.$$  \[2\]

Hence, $R - r = R \geq \sqrt{2} > 1 \geq \alpha_p$ (see Theorem 5.2).

Now consider the Weber problem with rectangular distance. We have already shown that the planar problem has the rounding property (see Lemma 3.2 ) which is easily transferred to $n$-dimensional problems using Theorem 4.1 as these problems have cross-shaped level sets.

Lemma 6.1: The nonempty level sets of the objective function of the $n$-dimensional Weber problem with rectangular distance,

$$f(x) = \sum_{m=1}^M w_m ||A_m - x||_1$$

are cross-shaped with respect to any optimal solution for this problem.
Proof: As seen above we can rewrite the function as \( f(x) = \sum_{i=1}^{n} f_i(x_i) \) where \( f_i(x_i) = \sum_{m=1}^{M} w_m |A_{mi} - x_i| \). As those \( f_i \) are convex \( f \) is separable convex. Thus the proposition follows from Lemma 4.2.

The third example in Section III was the centroid problem: we showed that its objective \( f: \mathbb{R} \rightarrow \mathbb{R}^2 \), \( X \mapsto \sum_{m=1}^{M} w_m ||A_m - X||_2^2 \) has the rounding property (1) (see Lemma 3.3).

We also mentioned that we can rewrite the objective (for arbitrary \( n \) dimensions) as \( \sum_{m=1}^{M} w_m (A_{m,i} - X_i)^2 \).

Since \( \sum_{m=1}^{M} w_m (A_{m,i} - X_i)^2 \) is convex for every \( i = 1, \ldots, n \) (recall that \( w_m \geq 0 \)) we know from Lemma 4.2 that the nonempty level sets of the function \( \sum_{m=1}^{M} w_m ||A_m - X||_2^2 \) are cross-shaped with respect to any optimal solution for the continuous centroid problem. Analogously we can prove that the objective function of the \( n \)-dimensional centroid problem has the rounding property. In [6] (p. 50) it is shown that the level sets are Euclidean balls, hence they are in this example also quasi-round.

Our concept of the rounding property is not limited to separable convex functions as is e.g. shown by \( f(x) = ||x||_2 \): the level sets \( L \leq (z) \) of this function are balls around 0 with radius \( z \) and thus surely cross-shaped with respect to 0 which is the continuous minimum. With Theorem 4.1 it follows that \( f \) has the rounding property (1). But it is also obvious that \( f \) is not separable convex.

Of course there are many other examples in which the rounding property holds. In Figure 7 we see an example for a one-dimensional function with quasi-round level sets w.r.t. \( x^* \) that has two specialties: the function is not continuous and the level sets are not symmetric w.r.t. \( x^* \).

Note that the concept of quasi-roundness also allows level sets that are not even connected.

VII. Extensions

In this section we show how we can extend our approach. We present two totally different ways: the first one is a more general way of defining the rounding property and the second one is a generalization of Theorem 4.1 and Theorem 5.1.

A. The specific rounding property

The definition of the rounding property might sometimes be too restrictive. Consider for example Figure 8.

The dark gray area is the set of optimal solutions, the light gray domain is (together with the dark gray one) the level set \( L \leq (z) \) for some other \( z \leq \bar{z} \). It holds that \( R - r \leq \alpha \) and so the level set is quasi-round w.r.t. \( x^*_1 \). But \( x^*_2 \) is also an optimal solution and \( R' - r' > \alpha \). Thus the level set is not quasi-round w.r.t. \( x^*_2 \). Similar examples can also be constructed for the level sets of the classical Weber problem.

Therefore it might be interesting to ask for the level sets to be quasi-round w.r.t. only one optimal solution.

Definition 7.1: We say that a function \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) has the specific rounding property if the following holds: There exists an optimal
solution $x^*$ to $(CP)$ \(\min \{ f(x) : x \in \mathbb{R}^n \} \) such that there is an optimal solution $x'$ of $(IP)$ \(\min \{ f(x) : x \in \mathbb{Z}^n \} \) with \(x' \in \text{Round}(x^*)\).

The advantage of this definition is that it is less restrictive than Definition 1 and so it is satisfied by a larger class of functions. If a function $f$ has the the rounding property (1) then it certainly also has the the specific rounding property (5). Moreover, both definitions are equivalent if $(CP)$ has only one optimal solution. The disadvantage is that it is no longer sufficient to find an arbitrary optimal solution of the continuous version but we have to find a specific one, namely one that gives the function $f$ the required property. This means in particular that $(IP)$ does not have to be polynomial solvable just because $(CP)$ is. But it still holds that if you have an optimal solution for $(CP)$ with the required property than you can solve $(IP)$ polynomially for any fixed dimension.

Of course it is also possible to apply this extension to the strong rounding property (2) in exactly the same way.

**B. Basic theorem**

Another possibility of extending the preceding results is to state which assumption we really need in Theorems 4.1 and 5.1 in general.

**Theorem 7.1:** Assume that for any optimal solution $x^*$ of $(CP)$ \(\min \{ f(x) : x \in \mathbb{R}^n \} \), there is no $z \in \mathbb{R}$ with $L_\leq(z) \cap \text{Round}(x^*) = \emptyset$ but $L_\leq(z) \cap \mathbb{Z}^n \neq \emptyset$. Then $f$ has the rounding property (1).

**Proof:** Choose any optimal solution $x^*$ of $(CP)$. Let $x \in \mathbb{Z}^n$ be an arbitrary point with objective value $z = f(x)$ and let $\bar{x} = \arg\min \{ f(x) : x \in \text{Round}(x^*) \}$, thus $f(\bar{x}) = \bar{z}$.

Our goal is to show that $f(\bar{x}) \leq f(x)$.

Case 1: $\bar{z} \leq z \Rightarrow f(\bar{x}) \leq f(x)$ or

Case 2: $\bar{z} > z$: In this case $\bar{x} \notin L_\leq(z)$ and therefore $L_\leq(z) \cap \text{Round}(x^*) = \emptyset$. But as $x \in L_\leq(z)$ we also have $L_\leq(z) \cap \mathbb{Z}^n \neq \emptyset$, which is not allowed due to our assumption.

This means that the second case leads to a contradiction and only the first case can happen and thus for all $x \in \mathbb{Z}^n$ it holds that $f(\bar{x}) \leq f(x)$. Therefore $\bar{x} \in \text{Round}(x^*)$ is an optimal solution for $(IP)$ and $f$ has the rounding property (1).

Of course this theorem holds since we made the assumption just the way it “fits”. But even if it is algorithmically not very helpful it shows the general idea we followed in this paper. Therefore it can be used to develop new forms of the level sets (like cross-shapedness oder quasi-roundness) that guarantee this assumption and hence lead to the rounding property.

Cross-shapedness and quasi-roundness are on the other hand special cases of this property as both guarantee that there is no $z \in \mathbb{R}$ that fulfills $L_\leq(z) \cap \text{Round}(x^*) = \emptyset$ and $L_\leq(z) \cap \mathbb{Z}^n \neq \emptyset$.
VIII. CONCLUSION AND FURTHER RESEARCH

In this paper we have used geometric considerations in order to identify functions that have the rounding property, i.e. functions that guarantee that we can find an optimal solution for the integer problem by solving the continuous problem and then comparing only the integer points we reach by rounding up and down the components of this continuous optimal solution. This has the advantage that the complexity of the problem is (more or less) given by the complexity of the continuous problem.

We currently investigate how constraints can be integrated into our concept in order to be able to solve more general nonlinear integer programming problems. Note that in case that $\text{Round}(x^*) \subseteq S$ where $S$ is the feasible region, our results still hold. A generalization is also possible in the case of a cross-shaped feasible region and an objective function whose level sets are cross-shaped with respect to the same point as the feasible region. In this case we can use that the intersection of two sets that are cross-shaped w.r.t. the same point is again cross-shaped w.r.t. this point. Other points are under research.

Another interesting question is how we can apply our results to mixed integer problems. This means we consider problems of the following type:

\[
\begin{align*}
\text{(MIP)} & \quad \min \{ f(x, y) : (x, y) \in S \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \}.
\end{align*}
\]

Take for example (as in Lemma 5.1) a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose level sets are balls (for example $f(x) = ||x - x^*||_2$) with the additional constraint $x_1 \in \mathbb{Z}$ (see Figure 9). We see that for the optimal solution $\bar{x}$ of

\[
\begin{align*}
\text{(MIP)} & \quad \min \{ ||x - x^*||_2 : x \in \mathbb{Z} \times \mathbb{R} \}
\end{align*}
\]

it holds that $\bar{x} = (\text{round}(x_1^*), x_2^*)$. This means that also here we get an optimal solution for (MIP) by rounding the solution for the continuous counterpart but with the difference that we round only the coordinates that are required to be integer.

---

APPENDIX

First Appendix

**Theorem 5.2** For $p \in \mathbb{N}$ it holds that $\alpha_p = \min \{ \alpha'_p, 1 \}$ with $\alpha'_p$ the optimal objective value of

\[
\begin{align*}
\text{(P') } & \quad \min \left\{ \sqrt{\sum_{i=1}^{n} a_{i}^p + (1 + a_1)^p - a_1^p} \right\}
\end{align*}
\]

s.t. $a_1 \leq a_j \quad \forall j \in \{2, \ldots, n\}$

$0 \leq a_i \leq 0.5 \quad \forall i \in \{1, \ldots, n\}$.

**Proof:** We can formulate $(P_p) =: (P)$ as follows

\[
\begin{align*}
\text{(P)} & \quad \min b(x) - a(x)
\end{align*}
\]

s.t. $x \in \mathbb{R}^n$

with

\[
\begin{align*}
\text{(P'_b)} & \quad b(x) = \min ||x - y||_p
\end{align*}
\]

s.t. $y \in \mathbb{Z}^n \setminus \text{Round}(x)$.
and

\[
(P_a) \quad a(x) = \min_{y \in \mathbb{Z}^n} \|x - y\|_p
\]

**Part 1: \( x \in \mathbb{Z}^n \)**

For \( x \in \mathbb{Z}^n \) we see that \( \alpha = 1 \). Therefore let \( x \notin \mathbb{Z}^n \) in the following. Then \( \text{Round}(x) \neq \{x\} \).

**Part 2: Reformulate \((P_a)\)**

Due to Lemma 5.1 we have

\[
\tilde{x} = \text{argmin}_{y \in \mathbb{Z}^n} \|x - y\|_p \in \text{Round}(x).
\]

This means

\[
a(x) := \min_{y \in \mathbb{Z}^n} \left( \sum_{i=1}^{n} (x_i - y_i)^p \right)^{1/p}
\]

**Part 3: Limit the feasible set of \((P_b)\):**

We can limit the feasible set of \((P_b)\) to the points

\[
B = \{ y \in \mathbb{Z}^n : y_j \notin \{[x_j] - 1, [x_j] + 1\} \text{ for one } j \in \{1, \ldots, n\} \text{ and } y_i \in \{[x_i], [x_i]\} \forall i \in \{1, \ldots, n\} \setminus \{j\} \}
\]

**Proof.** Let \( z \in \mathbb{Z}^n \setminus (\text{Round}(x) \cup B) \). Then \( z_i \in \{a \in \mathbb{Z} : a \leq [x_i]\} \cup \{a \in \mathbb{Z} : a \geq [x_i]\} \) for all \( i = 1, \ldots, n \). As \( z \notin \text{Round}(x) \) there is at least one \( j \in I = \{i : z_i \leq [x_i] - 1 \lor z_i \geq [x_i] + 1\} \).

Furthermore \( z \notin B \) and therefore there are at least two \( j_1, j_2 \in I \). Define \( k = \text{argmin}_i \{i \in I\} \) and:

\[
y_i := \begin{cases} [x_i] & \text{if } z_i \leq [x_i] \\ [x_i] & \text{if } z_i \geq [x_i] \end{cases} \text{ for } i \neq k,
\]

\[
y_k := \begin{cases} [x_k] - 1 & \text{if } z_k \leq [x_k] - 1 \\ [x_k] + 1 & \text{if } z_k \geq [x_k] + 1. \end{cases}
\]

Then \( |x_k - y_k| \leq |x_k - z_k|, |x_j - y_j| < |x_j - z_j| \) for all \( j \in I \setminus \{k\} \) (thus for at least one \( j \)) and \( |x_i - y_i| \leq |x_i - z_i| \) for all other \( i \in \{1, \ldots, n\} \setminus I \). Therefore \( \|x - y\|_p < \|x - z\|_p \).

\[\Box\]

So we have

\[
(P_b) \quad b(x) = \min_{y \in B} \|x - y\|_p
\]

**Part 4: Reformulate \((P_b)\)**

\[
\|x - y\|_p = \left( \sum_{i=1}^{n} (x_i - y_i)^p \right)^{1/p}
\]

As \( y_1 \in \{[x_1] - 1, [x_1], [x_1] + 1\} \), we only have the following four possibilities for the absolute value of the first summand:

\[
|x_1 - y_1| = \begin{cases} x_1 - [x_1] + 1 & \text{if } y_1 = [x_1] + 1 \\ x_1 - [x_1] & \text{if } y_1 = [x_1] \\ [x_1] - x_1 & \text{if } y_1 = 0 \\ [x_1] - x_1 + 1 & \text{if } y_1 = 1 \end{cases}
\]

For \( x_1 \in \mathbb{Z} : [x_1] = x_1 = [x_1] \)

\[a_1 = \min\{x_1 - [x_1], [x_1] - x_1\} = 0 \text{ and } |x_1 - y_1| = \begin{cases} 0 & \text{ if } y_1 = 0 \\ 1 & \text{ otherwise} \end{cases} \]

Let now \( x_1 \notin \mathbb{Z} \).

For \( a_1 = \min\{x_1 - [x_1], [x_1] - x_1\} = x_1 - [x_1] \) follows

\[
[x_1] - x_1 = [x_1] + 1 - x_1 = 1 - a_1.
\]

Hence

\[
|x_1 - y_1| = \begin{cases} x_1 - [x_1] + 1 & \text{if } y_1 = x_1 + 1 \\ x_1 - [x_1] & \text{if } y_1 = a_1 \\ [x_1] - x_1 = 1 - a_1 \\ [x_1] - x_1 + 1 = 2 - a_1. \end{cases}
\]

For \( a_1 = [x_1] - x_1 \) on the other hand follows

\[
x_1 - [x_1] = x_1 - [x_1] + 1 = 1 - a_1
\]
and so
\[ |x_1 - y_1| = \begin{cases} x_1 - \lfloor x_1 \rfloor + 1 = 2 - a_1 \\ x_1 - \lfloor x_1 \rfloor = 1 - a_1 \\ \lfloor x_1 \rfloor - x_1 = a_1 \\ \lfloor x_1 \rfloor - x_1 + 1 = a_1 + 1. \end{cases} \]

This means that in both cases we only have the same four possibilities. Analogously we only have four possibilities for \(|x_i - y_i|\) for \(i = 2, \ldots, n\):
\[ |x_i - y_i| = \begin{cases} a_i + 1 \\ a_i \\ 1 - a_i \\ 2 - a_i. \end{cases} \]

(For \(a_i = 0\) we only have the possibilities 1 or 0 (compare \((f)\)).)

But we know:
1. \(a_i \leq 0.5 \Rightarrow a_i \leq 1 - a_i\)
2. \(1 - a_i \leq a_i + 1\)
3. \(2 \cdot a_i \leq 1 \Rightarrow 1 + a_i \leq 2 - a_i\)

Together: \(a_i \leq 1 - a_i \leq a_i + 1 \leq 2 - a_i\) for all \(i \in \{1, \ldots, n\}\).

At the same time we can realize \(|x_i - y_i| = a_i\) and \(|x_i - y_i| = 1 - a_i\) only if \(y_i \in \{\lfloor x_i \rfloor, \lceil x_i \rceil\}\).

To minimize \(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\) we want to choose \(y \in B\) such that \(|x_i - y_i| = a_i\) for all \(i\). But this would include \(y \in \text{Round}(x)\). Therefore one summand \(|x_k - y_k|\) has to be “a little bit greater“: it cannot be \(1 - a_k\) because this would also include \(y_k \in \{\lfloor x_k \rfloor, \lceil x_k \rceil\}\).

Thus we choose \(|x_k - y_k| = 1 + a_k| for one \(k \in \{1, \ldots, n\}\) and for all other \(i\): \(|x_i - y_i| = a_i|.

(Minimizing does not give us \(|x_i - y_i| = 2 - a_i| or \(|x_i - y_i| = 1 - a_i\). This means we can assume that from now on we only have the possibilities \(|x_i - y_i| = a_i\) or \(|x_i - y_i| = a_i + 1\). For \(a_i = 0\) this leads to the alternatives \(|x_i - y_i| = 0\) or \(|x_i - y_i| = 1\) as seen above. Thus we do not have to differentiate any more whether \(a_i = 0\) or not.)

As a result we get for \(p \in \mathbb{N}\)
\[
\min_k \sqrt{\sum_{i \neq k} a_i^p + (1 + a_k)\min_k a_k^p}
\]
\[= \min_k \sqrt{\sum_{i=1}^{n} a_i^p + (1 + a_k)\min_k a_k^p}
\]
\[= \sum_{i=1}^{p} a_i^p + \min_k ((1 + a_k)\min_k a_k^p)
\]
\[(*) \]
\[\sum_{i=1}^{p} a_i^p + \min_k (1 + a_k)\min_k a_k^p - (\min_k a_k)^p
\]

Where \((*)\) follows because for \(p \in \mathbb{N}\) it holds that
\[ (1 + a_i)^p = \sum_{l=0}^{p} \binom{p}{l} a_i^l \]

Hence
\[ (1 + a_i)^p - a_i^p = \sum_{l=0}^{p-1} \binom{p}{l} a_i^l \]

and
\[ a_i \leq a_j \]
\[ \Rightarrow a_i^l \leq a_j^l \forall l \geq 0
\]
\[ \Rightarrow \sum_{l=0}^{p-1} \binom{p}{l} a_i^l \leq \sum_{l=0}^{p-1} \binom{p}{l} a_j^l
\]
\[ \Leftrightarrow (1 + a_i)^p - a_i^p \leq (1 + a_j)^p - a_j^p
\]

**Part 5: Simplify \((P)\)**

Therefore the optimal solution for the following program is also an optimal solution for \((P)\):
\[
\min \sum_{i=1}^{n} a_i^p + (1 + \min_k a_k)\min_k a_k^p - \min_k a_k^p
\]

s.t. \(a_i = \min\{x_i - \lfloor x_i \rfloor, \lceil x_i \rceil - x_i\}
\]
\[ \forall i \in \{1, \ldots, n\}
\]
\[ x \in \mathbb{R}^n\]

As \(x \in \mathbb{R}^n\) is an arbitrary point that influences the objective function only via the \(a_i\)’s
and as we are mainly interested in the optimal objective value, we can also solve the following program to find the optimal objective value:

\[
\min \left( \sqrt{\sum_{i=1}^{n} a_i^p + (1 + \min_k a_k)^p - (\min_k a_k)^p} - \sqrt{\sum_{i=1}^{n} a_i^p} \right)
\]

s.t. \(0 \leq a_i \leq 0.5\) \(\forall i \in \{1, \ldots, n\}\).

This means first we solve the problem without considering that the \(a_i\)’s depend on \(x\). Having solved this problem we can find a point \(x\) that fits to the optimal \(a_i\)’s.

Furthermore we can assume without loss of generality that \(a_1 \leq a_j\) for all \(j = 2, \ldots, n\). That way we have to solve problem \((P_p')\):

\[
\alpha_p' = \min \left( \sqrt{\sum_{i=1}^{n} a_i^p + (1 + a_1)^p - a_1^p} - \sqrt{\sum_{i=1}^{n} a_i^p} \right)
\]

s.t. \(a_1 \leq a_j\) \(\forall j \in \{2, \ldots, n\}\)

\(0 \leq a_i \leq 0.5\) \(\forall i \in \{1, \ldots, n\}\).

Together with Part 1 this gives us \(\alpha_p = \min\{\alpha_p', 1\}\).

**REFERENCES**


