Vehicle routing for mixed solid waste collection – comparing alternative hierarchical formulations

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Abstract—The aim of this paper is to present and compare alternative hierarchical formulations for the periodic vehicle routing problem applied to solid waste collection. The solution of this problem is a one-week plan of daily routes for the transportation of mixed solid waste from containers to disposal facilities, taking into consideration the frequency of collection of each container within the planning horizon, the road network and the resources available. The objective is to minimize operation costs.

The real-world case that supported this study was the collection of mixed solid waste in Ponte de Lima, a municipality from the north of Portugal. The problem was modelled as a Periodic Vehicle Routing problem (PVRP) with the additional constraint that routes must pass through one of the alternative disposal facilities before returning to the depot.

Based on this real case scenario, we propose a framework of MIP models with three hierarchical approaches besides the monolithic model. Some new estimates for downstream constraints were developed and integrated in the upstream levels in order to improve feasibility.

Keywords—Waste collection Hierarchical formulations Periodic vehicle routing

I. INTRODUCTION

The costs of the collection of solid waste range between 40 and 60% of a community’s solid waste management system expenditures [16]. An efficient management of the solid waste collection can therefore generate significant savings while ensuring hygiene patterns and satisfaction of the inhabitants, besides all the other advantages common to the efficient management of transportation systems.

This work is based on a real case concerning Ponte de Lima, a municipality in the north of Portugal. The municipality manages the collection of the mixed waste generated in Ponte de Lima and guarantees its transport to disposal facilities. The main objective of the work done with the municipality was the reduction of the collection costs, that are highly dependent of the distance travelled by the vehicles. The resources such as the number and location of the depots and containers, the number of vehicles and staff, as well as the collection frequency of the containers in each parish were already fixed.

The output of the study should therefore be the visiting calendar of each container within the weekly planning horizon, considering the constrains of the collection frequency, and the plan of the routes for each vehicle and day, with the additional constraint that the routes must go through a disposal facility to unload the waste before returning to the depot. Problems with these characteristics are modelled in the literature as Periodic Vehicle Routing Problems (PVRP), a variant of the Vehicle Routing Problem (VRP).

The PVRP is known to be an NP-hard problem and the additional constraints that had to be included to adapt the model to the real situation of Ponte de Lima made the resolution even more challenging. In order to be able to solve the real problem we built a framework with three hierarchical approaches, which we have tested along with the monolithic model. The hierarchical approaches are identified by the aggregation of the decisions in each level: (1) assign and route together; (2) assign days first - assign vehicles and route second; (3) assign first - route second and (4) assign days first - assign vehicles second - route third. Some estimates of downstream constraints were developed and added in upstream levels in order to guarantee feasibility. We compared the results obtained with the MIP formulations developed for the approaches and with the current practice of the municipality.

The remainder of this paper is organized as follows: in section 2, a brief review of the relevant literature is presented. The problem is described in section 3 and in section 4 the hierarchical framework as well as the developed formulations are presented. In section 5 the results obtained are described and the approaches compared. Conclusions are drawn in section 6.
II. Literature Review

Routing problems have been widely treated in the literature because of their high complexity and practical relevance. The Traveling Salesman Problem (TSP) is the most discussed routing problem and consists in determining a minimum distance route that begins in a given location, passes through all the other locations (customers) and returns to the initial location [15]. In the Vehicle Routing Problem (VRP), a fleet of vehicles with known capacity is available to visit customers which have a known demand. The objective is to design routes for the vehicles at minimal total cost, guaranteeing that all the customers are served and that the capacity of the vehicles is not exceeded [20]. This problem adds to the TSP the decision of which customers assign to which vehicles.

The Periodic Vehicle Routing Problem (PVRP) is an extension of the VRP where customers must be visited with pre-defined frequencies over an extended period. The additional component of the problem consists in the assignment of one visiting calendar from a given set to each customer. The overall objective is to assign routes to the vehicles for each day of the planning horizon that minimize the total travel cost. The visiting calendar of each client must be met and routes are subject to vehicle capacity and route duration constraints. This problem was formally introduced in 1974 by Beltrami and Bodin as a generalization of the VRP, precisely in an application of municipal waste collection [9].

Russel and Igo called the PVRP an “Assignment Routing Problem” and mentioned the difficulties of choosing a calendar for each customer together with solving the routing problem [9]. To deal with the complexity and large scale nature of the problem, several authors consider the PVRP as a multilevel problem:

1) In the first level, a calendar is selected for each customer. In this way, it is decided which customers are visited on each day of the planning horizon;
2) In the second level, and for each day of the planning horizon, customers are assigned to the vehicles available in that day;
3) Finally, in the third level, a route is designed for each combination of day and vehicle.

Note that in the VRP, only the last two decisions need to be made and over a single day only. Being the VRP an NP-hard problem, the PVRP is therefore at least as difficult [19].

A significant body of work has been evolving, with multiple variants, formulations and solution methods applied to the PVRP. Three important variants of the PVRP are mostly addressed in the literature: the PVRP with time window constraints – PVRPTW [7], with service choice – PVRP-SC [8], with multiple depots – MDPVRP [10] and with intermediate facilities – PVRP-IF [2]. In this last variant, capacity replenishment is possible at different points along the route. As far as formulations are concerned, the most used one is the 4-index formulation from Christofides and Beasley, based on the VRP 3-index formulation from Golden et al [9]. Other formulations have been emerging, considering only the assignment problems [18], [3], [14]. More recently, alternative modeling approaches have been emerging, such as the Set Partitioning (SP) [4]. For instances of realistic size, the problem has been solved mostly with heuristics and metaheuristics and in sequential phases. Two-phase solution methods are more commonly found (a survey on solution methods can be found in [9]).

[5] states that solving an hierarchical problem is more than solving a set of distinct problems. It is necessary to guarantee feasibility in the downstream levels by including approximate measurements of lower level constraints in upstream levels. In the PVRP, this means that in the assignment problems it is necessary to guarantee that the number of customers assigned to a vehicle in a day neither exceeds its capacity nor leads to subproblems where it is not possible to create any route without exceeding its maximum duration. Whereas vehicle capacity constraints have already appeared in assignment problems, approximate measurements of route duration have not been covered so far.

To conclude, and concerning waste collection, this practical application has already been studied in the literature, not only concerning mixed but also separate waste [21], [1], [19], [13], [17].

III. Problem Definition

The municipality of Ponte de Lima owns and operates a fleet of 5 vehicles with different capacities for the mixed-waste collection. These vehicles are parked in a garage in a central parish – Arca. The 994 mixed-waste containers are non-uniformly distributed over Ponte de Lima and the waste is periodically collected and transported to disposal facilities, where afterwards it is whether dumped in a controlled environment or transformed. The filling rates of the containers are highly dependent on the density of both the containers and the inhabitants of the region. They also depend on the collection frequency imposed. The collection is performed 6 days a week. Figure 1 shows the location of the two existing disposal facilities and the depot as well as the collection frequency of the containers within each parish.

Currently the plans are monthly hand-made, without
assuring that the collection frequency matches the frequencies defined for each parish.

A. Objective

Different filling rates led the municipality to establish different frequencies of collection for the containers. Therefore, for a given planning horizon, a set of routes is required for each vehicle as well as a visiting schedule for each container. Each route should consist of an ordered list of visiting sites that ends on a disposal facility to deposit the waste after collection. The lowest frequency for a container is one visit in a week, which suggests a collection plan of one week.

The objective is to minimize collection costs, which are essentially dependent on the distance traveled by the vehicles. Routes are constrained by vehicle capacity and work shift duration. Each container should be visited as many times per week as its frequency and the visiting days should be distributed as uniformly as possible through the period.

B. Problem Data

In this case study the municipality did not have much of the information needed, such as the exact location of the containers and the quantity of waste to collect from each container, which raised some difficulties.

In order to decrease the level of disruption with the current practice, deal with the lack of information and reduce the complexity of the problem, the 994 containers were aggregated by parish. This was possible because the municipality set the same frequency to all the containers of each parish. In fact, the distances between parishes are superior to the distances between containers (Ponte de Lima is a rural area) which reinforces the idea that containers in the same parish must all be visited on the same day and by the same vehicle. The visiting order within each parish is a decision of the driver.

A mapping software with a shortest path algorithm was used to estimate the distances between parishes. Therefore, distances are not euclidean. It was considered that the distance was the same independently of the direction. In fact, at this level of abstraction it is not relevant to take into consideration possible variations due to some rare one way streets.

Small models were developed to estimate the quantity to collect and the service duration in each collection site as well as the traveling time between sites. Firstly, and since the quantity of waste is distributed proportionally to the population, the quantity to collect was considered dependent of the waste produced per capita in Ponte de Lima, of the number of inhabitants and of the frequency of collection of each parish. Secondly, the service duration in each collection site depends on the number of containers in each parish and on the average time spent by container. Finally, traveling time is a function of the distance traveled and of the average speed of the vehicles.

For each visiting frequency a list of possible schedules was generated, trying to balance the time between consecutive visits. For instance, for a frequency of two visits a week, the following schedules were considered: {Monday, Thursday}, {Tuesday, Friday}, {Wednesday, Saturday}, {Monday, Friday} and {Tuesday, Saturday}.

Fig. 1. Ponte de Lima Collection System: (i) Disposal Facilities, (ii) Depot, (iii) Collection frequency in each parish
C. Notation

The waste collection problem will be formulated considering a direct graph $G = (\mathcal{V}, \mathcal{A})$, which represents the road network, and with the indices, parameters and sets of table I.

| $i, j, h$ | Location |
| $k$ | Vehicle |
| $l$ | Day |
| $r$ | Schedule |

| $L$ | Planning horizon (days) |
| $N$ | Number of parishes |
| $P$ | Number of possible disposal facilities |
| $K_i$ | Number of vehicles available on day $l$ |
| $d_{ij}$ | Distance between locations $i$ and $j$ (km) |
| $g_i$ | Total waste to be collected in parish $i$ (Kg) |
| $s_i$ | Service duration in parish $i$ (minutes) |
| $t_{ij}$ | Duration of route from $i$ to $j$ (minutes) |
| $Q_k$ | Capacity of vehicle $k$ (Kg) |
| $T_i$ | Maximum duration of a route on day $l$ (minutes) |
| $C_i$ | Number of allowable schedules for parish $i$ |
| $a_{rl}$ | Binary constant that equals one if day $l$ belongs to schedule $r$ |

| $\mathcal{L}$ | Days of the Planning Horizon, $\mathcal{L} = \{1, \ldots, L\}$ |
| $\mathcal{V}$ | Locations, $\mathcal{V} = \{v_0, v_1, \ldots, v_{N+p}\}$, where $v_0$ corresponds to the depot |
| $\mathcal{V}_c$ | Parishes, $\mathcal{V}_c = \{v_1, v_2, \ldots, v_N\}$ |
| $\mathcal{V}_p$ | Disposal facilities, $|\mathcal{V}_p| = P$, $\mathcal{V}_p = \{v_{N+1}, v_{N+2}, \ldots, v_{N+p}\}$ |
| $\mathcal{A}$ | Arcs, $\mathcal{A} = \{(v_i, v_j) : v_i, v_j \in \mathcal{V}, i \neq j\}$ |
| $K_l$ | Vehicles available on day $l$, $|K_l| = K_l$ |
| $C_i$ | Set of allowable schedules for parish $i$, $|C_i| = c_i$ |

IV. A FRAMEWORK OF ALTERNATIVE HIERARCHICAL FORMULATIONS

The problem described in section III can be formulated as a Periodic Vehicle Routing Problem. An additional constraint is observed though: routes must pass through a disposal facility to unload the waste before returning to the depot.

The decomposition of highly complex optimization problems into subproblems, hierarchically solved, is a well-known strategy in the literature (e.g. [14], [5]). Not only the problem becomes more efficiently solvable, but it is also taken into account that, in the context of real-world applications, these complex problems arise under broader decision making contexts, with decisions made by different actors and with different time horizon scopes. Therefore, it does make sense to break down the problem into subproblems, not loosing sight from the hierarchical relationships among them. On the other hand there is the well-known fact that solving until optimality a sequence of subproblems does not guarantee optimality for the overall problem resolution. However, given the size of real-world applications, the global optimum would be out of reach. An additional advantage of hierarchical approaches is the possibility of considering different optimization criteria at each level [14].

Bearing this in mind, in figure 2 we propose a framework of decomposition processes for the PVRP, based on different aggregations of the three decisions involved in the problem and identified in section II. In fact, the PVRP is too difficult to be solved directly by exact methods when considering instances of realistic size. All the subproblems identified are smaller and more amenable to rapid solutions.

The approaches are:

1) Deciding at the same time which customers will be served in each day of the week, by which vehicle, and in which sequence (assign and route together);

2) Deciding first which customers will be served in each day of the week, and afterwards by which vehicle and in which sequence (assign days first - assign vehicles and route second);

3) Deciding at the same time which customers will be served in each day of the week and by which vehicle, and afterwards in which sequence (assign first - route second);

4) Deciding first which customers will be served in each day of the week, then by which vehicle, and finally in which sequence (assign days first - assign vehicles second - route third).

![Fig. 2. Alternative Decomposition Approaches to the PVRP](image-url)
whereas in approaches 3 and 4 the TSP is solved a maximum of 30 times.

Some authors proposed approaches complementary to cluster first - route second, namely route first - cluster second. However, as [5] stated, these approaches do not perform as well from a computational perspective.

To build the framework, different formulations from the literature were put together, and divided by type of approach. All the problems identified in the framework were formulated taking into consideration the practical application features and the formulations scattered before. As far as routing is concerned, the traditional two (TSP) and three (VRP) index formulations were considered because of their greater flexibility in incorporating additional features [20]. To eliminate sub-tours, a transit load constraint was used instead of the traditional Dantzig-Fulkerson-Johnson subtour elimination constraint [20], [11], [15]. This constraint is a 4–index version of the generalized Miller-Tucker-Zemlin subtour elimination constraints instead of the traditional Dantzig-Fulkerson-Johnson (DFJ) subtour elimination constraints because of their polynomial cardinality. Originally proposed for the TSP by Miller, Tucker and Zemlin (MTZ) and later extended for the Asymmetric VRP, they are based on the vehicle load - the load increases when a vehicle passes through the customers and by summing up that value after each customer and assuring that it does not decrease, the route cannot return to an already visited customer. The only drawback is the much weaker LP

\[
\text{minimize } \sum_{i \in V} \sum_{j \in V; j \neq i} \sum_{l \in L} \sum_{k \in K_i} d_{ij} x_{ijkl} \quad (A.1)
\]

subject to

\[
\sum_{r \in C_i} y_{ir} = 1, \quad i \in V_c \quad (A.2)
\]

\[
\sum_{j \in V} \sum_{k \in K_i} x_{ijkl} - \sum_{r \in C_i} a_{rt} y_{ir} = 0, \quad i \in V_c; l \in L \quad (A.3)
\]

\[
\sum_{i \neq h} x_{ijkl} - \sum_{j \in V} x_{khjl} = 0, \quad h \in V \setminus \{v_0\}; l \in L; k \in K_l \quad (A.4)
\]

\[
u_{ikl} - u_{jkl} + Q_k x_{ijkl} \leq Q_k - q_j, \quad i, j \in V \setminus \{v_0\} : j \neq i; q_i + q_j \leq Q_k; l \in L; k \in K_l \quad (A.5)
\]

\[
q_i \leq u_{ikl} \leq Q_k, \quad i \in V \setminus \{v_0\}; l \in L; k \in K_l \quad (A.6)
\]

\[
\sum_{j \in V \setminus \{v_0\}} x_{0jkl} \leq 1, \quad l \in L; k \in K_l \quad (A.7)
\]

\[
\sum_{i \in V} \left( t_{ij} + s_i \right) x_{ijkl} \leq T_l, \quad l \in L; k \in K_l \quad (A.8)
\]

\[
\sum_{i \in V_p} \sum_{j \in V \setminus \{v_0\}} x_{ijkl} - \sum_{j \in V \setminus \{v_0\}} x_{0jkl} = 0, \quad l \in L; k \in K_l \quad (A.9)
\]

\[
x_{ijkl}, y_{ir} \in \{0, 1\}, \quad i, j \in V_c ; j \neq i; l \in L ; k \in K_l ; r \in C_i \quad (A.10)
\]

The objective function (A.1) minimizes the total distance traveled by the vehicles in all days of the planning period. Constraints (A.2) ensure that a feasible schedule is chosen for each customer whereas constraints (A.3) guarantee that customers are visited only on the days corresponding to the assigned schedules. Both constraints jointly determine the assignment of customers to the days of the planning period.

The following three sets of constraints guarantee a correct route design. Constraints (A.4) ensure connectivity by stating that when a vehicle arrives to a customer in a given day, it also leaves him. However, sub-tours might still happen and constraints (A.5)-(A.6) are subtour elimination constraints. We opted for the use of these constraints instead of the traditional Dantzig-Fulkerson-Johnson (DFJ) subtour elimination constraints because of their polynomial cardinality. Originally proposed for the TSP by Miller, Tucker and Zemlin (MTZ) and later extended for the Asymmetric VRP, they are based on the vehicle load - the load increases when a vehicle passes through the customers and by summing up that value after each customer and assuring that it does not decrease, the route cannot return to an already visited customer. The only drawback is the much weaker LP
relaxation provided by this family of constraints when compared with DFJ. However, it reduces significantly the number of constraints - the number of constraints in DFJ grows exponentially with the number of customers.

The remainder of the constraints concern the problem-oriented characteristics of the routes. Constraints (A.7) state that each vehicle is used at most once a day and also that routes must start on the depot. Limits on route duration are imposed in (A.8). MTZ constraints additionally impose capacity requirements and consequently, when using them, limits on vehicle capacity are already imposed. Finally, constraints (A.9) state that vehicles must pass through a disposal facility immediately before returning to the depot. In these constraints, this new feature is obtained by stating that if a vehicle leaves the depot, it needs to return through one of the alternative disposal facilities.

When comparing this formulation with the original one, the most relevant changes are the inclusion of an additional set of constraints - (A.9) and other minor changes to account with the existence of disposal facilities. Additionally, it was considered that vehicles might not be available in all days of the planning period. Another difference was the use of MTZ subtour elimination constraints. In spite of the assumption that distances are independent of the direction, we used the asymmetric version of the problem in order to use these subtour elimination constraints.

### B. Day and Vehicle Assignment Problem

In order to reduce the complexity of the PVRP, some formulations are presented in the literature which do not explicitly specify the routing constraints. Therefore, the solution to these problems is an assignment of customers to days and vehicles. These formulations differ mostly in the objective function.

The Day and Vehicle Assignment formulation is based on a tactical planning model developed by [14]. It is formulated in terms of decision variables \( x_{ikl} \), which equal 1 if customer \( i \) is visited in period \( l \) by vehicle \( k \), and variables \( y_{ir} \) which take the value 1 if schedule \( r \) is assigned to customer \( i \). Additionally, a penalty \( P_1 \) was introduced for each vehicle leaving the depot on each day. For that, the additional variables \( w_{kl} \) were defined, which equal one if any customer is assigned to vehicle \( k \) in period \( l \). In order to linearize the model, a set of variables \( z_{ijl} \) was also used, taking value one if customers \( i \) and \( j \) are both assigned to the same vehicle in period \( l \).

This formulation is presented below, ranging from equation (B.1) to (B.8).

\[
\min \sum_{i \in V_c} \sum_{j \in V_c} \sum_{l \in L} d_{ij} z_{ijl} + \sum_{l \in L} \sum_{k \in K_l} P_1 w_{kl} \tag{B.1}
\]

subject to

\[
\sum_{r \in C_c} y_{ir} = 1, \quad i \in V_c \tag{B.2}
\]

\[
\sum_{k \in K_c} x_{ikl} - \sum_{r \in C_c} a_{rl} y_{ir} = 0, \quad i \in V_c; l \in L \tag{B.3}
\]

\[
x_{ikl} + x_{jkl} - z_{ijl} \leq 1, \quad i, j \in V_c; j < i; l \in L; k \in K_l \tag{B.4}
\]

\[
\sum_{i \in V_c} q_i x_{ikl} \leq Q_k, \quad l \in L; k \in K_l \tag{B.5}
\]

\[
\sum_{i \in V_c} S_2(s_i + t_i) x_{ikl} = T_l - t_{out} - t_{in}, \quad l \in L; k \in K_l \tag{B.6}
\]

\[
\sum_{i \in V_c} x_{ikl} - N w_{kl} \leq 0, \quad l \in L; k \in K_l \tag{B.7}
\]

\[
z_{ijl}, y_{ir}, w_{kl}, x_{ikl} \in \{0,1\}, \quad i, j \in V_c; j < i; r \in C_c;
\]

\[
l \in L; k \in K_l \tag{B.8}
\]

In the objective function of equation (B.1), the first component minimizes the sum of the distances between all the customers assigned to the same vehicle on the same day. The additional set of variables \( z \) was used to linearize this function, which otherwise would be represented as in equation (B.9). For that purpose, constraints (B.4) aim to define \( z_{ijl} \) in terms of the values of \( x_{ikl} \) and \( x_{jkl} \).

\[
\sum_{i \in V_c} \sum_{j \in V_c} \sum_{k \in K_l} \sum_{l \in L} d_{ij} x_{ikl} x_{jkl} \tag{B.9}
\]

Still concerning the objective function, the number of distances to sum in each cluster increases considerably with the number of customers assigned to it. Therefore, customers tend to be balanced through the clusters, minimizing in this way the number of customers in each one. To counterbalance this fact, the second component of the objective function aims to penalize each route created. Otherwise, solutions would be homogeneous routes, and all the vehicles would be used in all days.

Concerning the other constraints, (B.2) and (B.3) guarantee a feasible assignment of customers to days and vehicles and have similar meanings as constraints (A.2) and (A.3) in the periodic vehicle routing problem formulation. Constraints (B.5) and (B.6) correspond to two downstream constraints: vehicle capacity limit and route maximum duration. Constraints (B.7) relate variables \( x_{ikl} \) with variables \( w_{kl} \).
In order to limit route duration in this upstream level, it was necessary to estimate the duration of each route. Along with the homogeneity within the clusters, the objective function also achieves separation of clusters. Each route can therefore be represented as in figure 3 (i) and it has three components: the traveling time of going from the depot to the cluster \( t_{out} \), the time spent visiting all the customers inside the cluster \( \sum_{i \in V_c} [S_2 \times (s_i + t_i)] x_{ikl} \) and finally the time to arrive to the disposal facility, discharge the waste and return to the depot \( t_{in} \). The problem lies in estimating each component once the customers assigned to each cluster are not known in advance.

![Cluster of Customers](image)

Fig. 3. Route duration estimation

The parameter \( t_{out} \) is the average between the time from the depot to the nearest and farthest customer. Similarly, \( t_{in} \) is the average between the time from the nearest and farthest customer to the depot. It includes the time spent in the disposal facility as well. Concerning the cluster, service time in each customer is known and only traveling times need to be estimated. Parameters \( t_i \) were defined representing the margin of contribution of each customer \( i \) to any route. These values were computed as shown in figure 3 (ii) and indicated in equation (B.10). They are based on the savings from the Clark and Wright heuristic [12] and represent the extra-time of the route due to the introduction of customer \( i \) between customers \( j \) and \( h \), where \( j \) and \( h \) are considered to be the two customers closest to \( i \) with the same frequency. \( S_2 \) is a parameter introduced to adjust the model to each specific problem.

\[
t_i = t_{ij} + t_{ih} - t_{jh} \quad \text{(B.10)}
\]

When comparing this new formulation with the original one, a new component of the objective function and a new set of constraints were introduced in order to penalize each vehicle that leaves the depot on each day. Computational experiments need to be performed in order to tune parameter \( P_1 \). A new downstream constraint was also introduced: route maximum duration. For that, it was necessary to create a form of estimating route duration without the visiting order of the customers. Finally, and since the distances are independent of the direction, we turned the model symmetric and only used half of the matrix of the decision variables \( x \).

C. Day Assignment Problem

Based on the Day and Vehicle Assignment formulation from section IV-B, and by not considering the assignment of customers to vehicles, we obtain a simpler formulation that only assigns each customer to an allowable schedule.

In the resulting Day Assignment formulation, the set of variables \( y \) captures all the decision making. \( y_{ir} \) equals one if customer \( i \) is visited by schedule \( r \). The decision variables \( x_{ikl} \) are no longer needed because by using \( \sum_{r \in C_i} a_{rl}y_{ir} \) we can obtain if customer \( i \) is assigned to a specific day \( l \). Similarly to the previous model, \( P_1 \) is used to penalize each vehicle leaving the depot on each day. For that, \( w_{kl} \) takes value one if any customer is assigned to vehicle \( k \) in period \( l \). The model is also linearized by the set of variables \( z_{ijl} \) which take value one if customers \( i \) and \( j \) are both assigned to the same vehicle in period \( l \).

This new formulation ranges from equation (B.1) to (B.8).

\[
\begin{align*}
\text{minimize} & \sum_{i \in V_c} \sum_{j \in V_c} \sum_{l \in L} d_{ij} z_{ijl} + \sum_{l \in L} \sum_{k \in K_l} P_1 w_{kl} \\
\text{subject to} & \sum_{r \in C_i} y_{ir} = 1, \quad i \in V_c \\
& \sum_{r \in C_i} a_{rl}y_{ir} + \sum_{r \in C_j} a_{rl}y_{jr} - z_{ijl} \leq 1, \quad i, j \in V_c : \ j < i; \quad l \in L \\
& \sum_{i \in V_c} q_{l}a_{rl}y_{ir} \leq S_1 \sum_{k \in K_l} Q_k w_{kl}, \quad l \in L \\
& \sum_{i \in V_c} S_2 (s_i + t_i) a_{rl}y_{ir} \leq \sum_{k \in K_l} (T_1 - t_{out} - t_{in}) w_{kl}, \quad l \in L \\
& z_{ijl}, y_{ir}, w_{kl} \in \{0, 1\}, \quad i, j \in V_c : j < i; r \in C_i; \quad l \in L; k \in K_l
\end{align*}
\]
which only ensures that the total quantity assigned to a day does not exceed the total vehicle capacity available on that day. Consequently, a parameter $S_1$ (with value less than 1) was introduced to guarantee that the solution does not split the quantity of a customer into multiple vehicles (for instance, being two vehicles of 40 kg available, a feasible solution could be two customers of 30 kg and a customer of 20 kg. However, this solution would lead to an infeasible problem in the next level).

Although this formulation does not include vehicle assignment, the number of vehicles used may be reduced by introducing the second component of the objective function, penalty $P_1$ and variable $w_{kl}$ defined in equation (C.4).

### D. Vehicle Assignment Problem

In the Vehicle Assignment Problem, it was considered that a schedule was already assigned to each customer, forming $t$ new sets containing the customers assigned to each day $l - \mathcal{V}_l \subset \mathcal{V}_e, l = 1, ..., t$. Still based on section IV-B, this formulation uses only decision variables $x$: $x_{ik}$ takes value one if customer $i$ is visited by vehicle $k$. $P_2$ penalizes each vehicle leaving the depot. For that, $w_{kl}$ takes value one if any customer is visited by vehicle $k$. Variables $z_{ij}$ are created to linearize the model and equal one if customers $i$ and $j$ are both assigned to the same vehicle.

For each independent day $l$ the problem is formulated by the model ranging from equation (D.1) to (D.7).

$$ \text{minimize} \sum_{i \in \mathcal{V}_l} \sum_{j \in \mathcal{V}_l, j < i} d_{ij} z_{ij} + \sum_{k \in \mathcal{K}_l} P_2 w_{kl} \quad \text{(D.1)} $$

subject to

$$ \sum_{k \in \mathcal{K}_l} x_{ik} = 1, \; i \in \mathcal{V}_l \quad \text{(D.2)} $$

$$ x_{ik} + x_{jk} - z_{ij} \leq 1, \; i, j \in \mathcal{V}_l : j < i; k \in \mathcal{K}_l \quad \text{(D.3)} $$

$$ \sum_{i \in \mathcal{V}_l} q_i x_{ik} \leq Q_k, \; k \in \mathcal{K}_l \quad \text{(D.4)} $$

$$ \sum_{i \in \mathcal{V}_l} x_{ik} - N w_{kl} \leq 0, \; k \in \mathcal{K}_l \quad \text{(D.5)} $$

$$ \sum_{i \in \mathcal{V}_l} S_2(s_i + t_i) x_{ik} \leq T_l - t_{out} - t_{in}, \; k \in \mathcal{K}_l \quad \text{(D.6)} $$

$$ z_{ij}, w_{kl}, x_{ik} \in \{0, 1\}, \; i, j \in \mathcal{V}_l : j < i; k \in \mathcal{K}_l \quad \text{(D.7)} $$

The day assignment constraints (B.2) and (B.3) were removed and constraint (D.2) was introduced to state that each customer must be assigned to one of the available vehicles. The remaining constraints are similar but considering only day $l$. Note that when computing $t_i$ through equation (B.10), since the customers assigned to each day are already known, $j$ and $h$ represent the two closest customers to $i$ on that day.

### E. Vehicle Routing Problem

When the planning horizon is a single day ($L = 1$), the PVRP reduces to a VRP. Therefore, the VRP formulation ranging from equation (E.1) to (E.9) is similar to the PVRP formulation but without the day assignment features. Due to the fact that this problem is going to be applied after day assignment, the only major change is the use of the sets $\mathcal{V}_l \subset \mathcal{V}$ and $\mathcal{V}_l \subset \mathcal{V}_e$, containing the locations assigned to day $l$ (with and without depot).

One set of variables contains the overall solution of the decision process: $x_{ijk}$, which takes value one if vehicle $k$ visits customer $j$ immediately after visiting customer $i$. The additional variables $u_{jk}$ for subtour elimination constraints represent the load of vehicle $k$ after visiting customer $i$.

For each independent day $l$ the problem is formulated by the model ranging from equation (E.1) to (E.9).

$$ \text{minimize} \sum_{i \in \mathcal{V}_l} \sum_{j \in \mathcal{V}_l, j \neq i} \sum_{k \in \mathcal{K}_l} d_{ij} x_{ijk} \quad \text{(E.1)} $$

subject to

$$ \sum_{i \in \mathcal{V}_l} \sum_{j \in \mathcal{V}_l, j \neq i} x_{ij} = 1, \; j \in \mathcal{V}_l \quad \text{(E.2)} $$

$$ \sum_{i \in \mathcal{V}_l} \sum_{j \in \mathcal{V}_l, j \neq i} x_{hjk} = 0, \; h \in \mathcal{V}_l \setminus \{v_0\}; k \in \mathcal{K}_l \quad \text{(E.3)} $$

$$ u_{ik} - u_{jk} + Q_k x_{ijk} \leq Q_k - q_j, \; i, j \in \mathcal{V}_l \setminus \{v_0\} : j \neq i; \quad \text{(E.4)} $$

$$ q_i \leq u_{ik} \leq Q_k, \; i \in \mathcal{V}_l \setminus \{v_0\}; k \in \mathcal{K}_l \quad \text{(E.5)} $$

$$ \sum_{j \in \mathcal{V}_l \setminus \{v_0\}} x_{0jk} \leq 1, \; k \in \mathcal{K}_l \quad \text{(E.6)} $$

$$ \sum_{i \in \mathcal{V}_l \setminus \{v_0\}} \sum_{j \in \mathcal{V}_l \setminus \{v_0\}} (t_{ij} + s_i) x_{ijk} \leq T_l, \; k \in \mathcal{K}_l \quad \text{(E.7)} $$

$$ \sum_{i \in \mathcal{V}_p} x_{ij0} - \sum_{j \in \mathcal{V}_l \setminus \{v_0\}} x_{0jk} = 0, \; k \in \mathcal{K}_l \quad \text{(E.8)} $$

$$ x_{ijk} \in \{0, 1\}, \; i, j \in \mathcal{V}_l : j \neq i; k \in \mathcal{K}_l \quad \text{(E.9)} $$

Constraints (E.2) impose that each customer is only visited once. Constraints (E.3), (E.4) and (E.5) assure
a correct route design. Note the use of the MTZ family of subtour elimination constraints. Finally, problem-oriented characteristics of the routes are guaranteed through constraints (E.6)-(E.8). Similarly to the PVRP formulation, limits on vehicle capacity are set by the subtour elimination constraints.

**F. Traveling Salesman Problem**

This classical formulation of the Traveling Salesman Problem is asymmetric and has as decision variables the set \( x_{ij} \) takes value one if customer \( j \) is visited immediately after \( i \). Similarly to the last formulations, this one is applied after a previous assignment of customers to days and vehicles. Therefore, the following sets were used: \( V_{kl} \subset V \) and \( V_{c}\ell k \subset V_c \) which contain the locations assigned to day \( l \) and vehicle \( k \) (with and without depot). Variables \( u_i \) define the order in which each customer \( i \) is visited for subtour elimination.

For each independent day \( l \) and vehicle \( k \) the problem is formulated by the model ranging from equation (F.1) to (F.9).

\[
\text{minimize} \quad \sum_{i \in V_{kl}} \sum_{j \in V_{ki}, j \neq i} d_{ij} x_{ij} \quad \text{(F.1)}
\]

subject to

\[
\sum_{j \in V_{ki}} x_{ij} = 1, \quad i \in V_{c\ell k} \quad \text{(F.2)}
\]

\[
\sum_{i \in V_{pl}} x_{ij} = 1, \quad j \in V_{c\ell k} \quad \text{(F.3)}
\]

\[
\sum_{i \in V_p} x_{ij0} = 1 \quad \text{(F.4)}
\]

\[
\sum_{i \in V_{kl}} x_{ij} = \sum_{j \in V_p} x_{ij0} = 0, \quad j \in V_p \quad \text{(F.5)}
\]

\[
\sum_{i \in V_{kl}} \sum_{j \in V_{ki}, j \neq i} (s_i + t_{ij}) x_{ij} \leq T_l \quad \text{(F.6)}
\]

\[
u_i - u_j + (N + 1)x_{ij} \leq N, \quad i, j \in V_{kl}\setminus\{v_0\} : j \neq i \quad \text{(F.7)}
\]

\[
1 \leq u_i \leq N + 1, \quad i \in V_{kl}\setminus\{v_0\} \quad \text{(F.8)}
\]

\[
x_{ij} \in \{0, 1\}, \quad i, j \in V_{kl} \quad \text{(F.9)}
\]

The objective function (F.1) minimizes the total distance of the route. Constraints (F.2) and (F.3) state that the route must arrive and leave every customer. Together with constraints (F.5), (F.7) and (F.8) they guarantee the connectivity of the route. These two last constraints are the MTZ subtour elimination constraints. The maximum duration of the route is guaranteed by constraint (F.6) and finally, constraints (F.4) state that the last visited locations must be a disposal facility. [15] defined this asymmetric TSP basic formulation as an assignment problem with additional constraints.

**V. Computational Results**

The alternative approaches, and corresponding MIP formulations, were evaluated with the case study instance, whose characteristics were described in section III. Preliminary computational experiments were carried out to define values for the parameters of each formulation. The results were compared in terms of objective function value (TD), total execution time (ET) and average gap between the integer solution and the lower bound found by CPLEX in each sub-problem (Gap). Additionally, the number of routes (NR) and the duration of the longest route (TT) were recorded.

The individual problems were solved through the IBM ILOG CPLEX 12.1 optimizer and were integrated in a Visual Studio 2008 project, under a C++ environment. The tests were performed on a Linux workstation equipped with an Intel XEON Dual Core 5160, 3GHz. The running time of each individual problem was limited. These maximum running times were set so that the global running time for every approach was equal to 4500 seconds. Those values are presented in table II.

**TABLE II
Maximum execution times used in the test instance**

<table>
<thead>
<tr>
<th>Approach</th>
<th>( TL_1 ) (s)</th>
<th>( NP_1 )</th>
<th>( TL_2 ) (s)</th>
<th>( NP_2 )</th>
<th>( TL_3 ) (s)</th>
<th>( NP_3 )</th>
<th>Total (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4500</td>
<td>1</td>
<td>4500</td>
<td>1</td>
<td>4500</td>
<td>1</td>
<td>4500</td>
</tr>
<tr>
<td>2</td>
<td>900</td>
<td>1</td>
<td>600</td>
<td>6</td>
<td>4500</td>
<td>1</td>
<td>4500</td>
</tr>
<tr>
<td>3</td>
<td>2700</td>
<td>1</td>
<td>60</td>
<td>30</td>
<td>4500</td>
<td>1</td>
<td>4500</td>
</tr>
<tr>
<td>4</td>
<td>900</td>
<td>1</td>
<td>300</td>
<td>6</td>
<td>60</td>
<td>30</td>
<td>4500</td>
</tr>
</tbody>
</table>

\( TL_x \) - time limit in level \( x \); \( NP_x \) - number of subproblems of level \( x \)

Table III reports the results obtained for the case study instance, after having the formulations’ parameters \( S_1, S_2, P_1 \) and \( P_2 \) tuned in the preliminary computational experiments.

By analyzing the results, it is possible to see that, due to the complexity of the PVRP, the monolithic model did not achieve any solution within the time limit. The best results were obtained with approach 2 (assign days first - assign vehicles and route second), not only concerning total distance but also the number of routes. With a solution containing 100km and one route more, the approach 4 performed in second (assign days first, assign vehicles second, route third).
TABLE III
TEST INSTANCE RESULTS

<table>
<thead>
<tr>
<th>Approach</th>
<th>(S_1)</th>
<th>(S_2)</th>
<th>(P_1)</th>
<th>(P_2)</th>
<th>#NR</th>
<th>TD (km)</th>
<th>TT (s)</th>
<th>ET</th>
<th>Gap_1</th>
<th>Gap_2</th>
<th>Gap_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>4500</td>
<td>4577.6</td>
<td>410.82</td>
<td>2715</td>
<td>39%</td>
<td>67%</td>
<td>0%</td>
</tr>
<tr>
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<td>1.00</td>
<td>1.15</td>
<td>0</td>
<td>28</td>
<td>2232.4</td>
<td>402.57</td>
<td>2715</td>
<td>39%</td>
<td>67%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>3</td>
<td>1.10</td>
<td>200</td>
<td>28</td>
<td>1883.7</td>
<td>383.34</td>
<td>2715</td>
<td>39%</td>
<td>0.36%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>4</td>
<td>0.95</td>
<td>1.15</td>
<td>200</td>
<td>27</td>
<td>1883.7</td>
<td>383.34</td>
<td>2715</td>
<td>39%</td>
<td>0.36%</td>
<td>0%</td>
<td>0%</td>
</tr>
</tbody>
</table>

\(S_1, S_2, P_1\), and \(P_2\) - formulations parameters; \#NR - number of routes; TD - total distance; TT - maximum traveling time; ET - execution time; Gap_x - average gap in level x; * - no solution

The fundamental reason for decomposing a problem is that the overall problem is too difficult to be solved monolithically. Thus, it is essential that individual problems are efficiently solvable. On the other hand, when increasing the number of levels we are restricting more and more the solution space. These facts, supported by the results obtained, raise once more the question of the trade–off between the number of decompositions and the difficulty of the resulting problems.

VI. CONCLUSIONS

In this paper, motivated by a real case scenario of a waste collection problem, we proposed a framework of MIP models with a monolithic model and three hierarchical approaches to the Periodic Vehicle Routing Problem. The hierarchical approaches were identified by the aggregation of the decision variables in each level: (1) assign and route together; (2) assign days first - assign vehicles and route second; (3) assign first - route second and (4) assign days first - assign vehicles second - route third. Estimates of downstream constraints were also developed and added at the upper levels in order to improve feasibility at the lower levels: maximum duration of routes and maximum load capacity of vehicles.

The hierarchical approach (2), assign days first - assign vehicles and route second, led to better results considering not only the total distance traveled but also the total number of routes. The hierarchical resolution raised an important point: the trade–off between the number of decompositions and the difficulty of the resulting subproblem.

As future work, the framework can be extended to take into account multiple depots (MDPVRP) and other MIP formulations might be developed for the subproblems. Another area of future research is the incorporation of other optimization criteria.

ACKNOWLEDGEMENTS

The first author is grateful to the Portuguese Foundation for Science and Technology for awarding her the grant SFRH/BD/74387/2010.

REFERENCES


